INTERDISCIPLINARY MATHEMATICS
VOLUME IX

GEOMETRIC STRUCTURE THEORY OF SYSTEMS—CONTROL THEORY AND PHYSICS, PART A

ROBERT HERMANN

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PREFACE

In this Volume, I begin the main *scientific* work of this treatise - the description of the differential geometric and algebraic tools and structure which describe and interrelate diverse areas of physics and engineering.

First, I would like to distinquish my use of the word "mathematical structure" from the way it is used by Bourbaki. I mean it more in the sense it is used in such branches of algebra as group theory - the study of the ways of decomposing complicated systems into simpler ones, and the classification of the simplest building blocks.

It may seem surprising to the reader that such algebraically-motivated concepts are of importance in Geometry and in the areas of application which depend on geometry. In fact, such a viewpoint is (like so much else) implicit in much of E. Cartan's work. (This is no doubt influenced by his first famous work, on the structure of Lie groups and Lie algebras.)

What has this to do with applications? I claim that often when mathematics is *useful* in an application it is because certain structure problems of this sort are tractable. Here are some examples:

a) Galois Theory and the solvability of equations.

b) Lie theory of differential equations and groups. The "reason" certain types of differential equations are useful *and* solvable is that they admit certain groups of automorphism (or "symmetries" as they are called in physics).

c) Linear, stationary input-output systems, and the associated prediction-control theory.

d) Quantum mechanics and relativity.

e) Certain aspects of elementary particle physics, e.g. unitary symmetry.

Now, I do not claim that structural relations are central to the *discovery* of new facts - it is in fact very rare. (Although perhaps one can see at least a trace of it in the work of the great (19th Century) Masters, e.g. Gibbs, Maxwell, Poincaré.) Its value to the contemporary scientific world lies rather in its *potential* ability to synthesize ideas in widely varying scientific disciplines. Thus, the similiarities between, say, physics and economics lie in certain common mathematical structural features. This has been pointed out most eloquently by Samuelson, who warns, however that it is not a question of finding a literal meaning in economics of physical concepts (e.g. "entropy"), but of an influence based on the use of certain common mathematical tools (e.g. optimization) and scientific poetry and analogy (e.g. ideal gases and Competitive Equilibrium). More concretely, I believe that there are tremendous scientific potentialities in a synthesis of engineering systems theory and physics, and use of the resulting intellectual tools in the even more complicated problems of biology and social behavior.

This volume is to a certain extent a continuation of "Geometry, physics and systems." In fact, the first few chapters contain material that I meant to include in Volume II of that book. After this basic material, the emphasis changes to applications of the differential geometric ideas in that broad interdisciplinary area called "Systems Theory." I have included certain algebraic material concerning linear ordinary differential equations (algebra of boundary conditions and the Kronecker algebraic structure theory of first order constant coefficient systems) because it illustrates my point that a *combination* of algebraic and geometric ideas is often needed. Further, I hope ultimately to generalize these results to *non-linear* and *partial differential equations*, perhaps in conjunction with Cartan's theory of exterior differential systems.

Next, I focus on the notion of *interaction* in various disciplines. I discuss a common viewpoint, closely related to the geometric properties of *differential forms*. In a later Volume, I hope to show that "interaction" in economics can be described in a analogous way.

The notations and conventions are much the same as in previous Volumes in this series. Familiarity with manifold theory is taken for granted. My previous books are referred to by abbreviations listed in the Bibliography, e.g., "Differential geometry and the calculus of variations" and "Geometry, physics and systems" (which are the basic references for differential-geometric notations) are DGCV and GPS.

I would like to thank Roger Brockett, Paul Fuhrmann, George Oster, John Stachel and Hector Sussman. Alta Zapf has again helped me immeasurably with typing.

INTERDISCIPLINARY MATHEMATICS VOLUME IX

Geometric Structure Theory of Controls,
Systems and Circuits, Part A

Contents

Chapter I

LIE'S THEORY OF GROUPS AND DIFFERENTIAL EQUATIONS

The evolution of what is now known as the theory of Lie groups is a strange phenomenon in the history of mathematics. It would take a full scale historical study to sort out this story, and I will not go into it in any detail here. What I find most remarkable however is that the theory first arose in Lie's mind in close contact with what are now known as "groups of symmetries of differential equations" - and with the theory of differential equations itself - and almost all of the work by mathematicians in this area up to the 1920's was oriented in this direction. Suddenly, in response to the increasing abstraction and algebrization of mathematics, attention switched to abstract group theory, and the roots from which the theory arose were forgotten. In this chapter, I plan to describe some of those roots.

1. GROUPS DEFINED BY DIFFERENTIAL EQUATIONS

Let M be a manifold. For $p \in M$, and each integer $r \geq 1$, let M_p^r denote the space of r-th order tangent vectors at the point p. (See Pohl, [1], and the brief exposition of this concept in GPS, Chapter 3.) Each map ϕ: M → M determines a linear map ϕ_*: $M_p{}^r \to M^r{}_{\phi(p)}$ called the <u>differential</u> of ϕ at p. Two maps ϕ, ϕ': M → M are said to <u>agree to the r-th order at the point p ∈ M</u> if:

$$\begin{aligned} &\phi(p) = \phi'(p) \\ &\phi_*(v) = \phi_*'(v) \end{aligned} \qquad (1.1)$$

for all $v \in M_p{}^r$.

Let G(M) denote the space of all diffeomorphisms ϕ: M → M. Then, G(M) forms a group, with the group-law just the composition of mappings. Let us apply the jet-construction to G(M).

Form G(M) × M. Introduce in it an equivalence relation:

(ϕ, p) is equivalent to (ϕ', p') if p = p';
ϕ and ϕ' agree to the r-th order at p.

Denote by $J^r(G(M))$ the quotient of M × G(M) by this equivalence relation. This space is called the space of <u>invertable r-jets of M</u>. Given $\phi \in G(M)$, let $j^r(\phi)$ be the r-jet of ϕ, considered as a map: M → $G^r(M)$. For p ∈ M, $j^r(\phi)(p)$ is the equivalence class to which (p, ϕ) belongs.)

Define a projection map π: $J^r(G(M)) \to M \times M$ as follows:

$$\pi(j^1(\phi)(p)) = (p, \phi(p)) \qquad (1.2)$$

for $\phi \in G(M)$, $p \in M$.

For $(p, p') \in M \times M$, set:

$$J^r(p, p') = \pi^{-1}(p, p') \tag{1.3}$$

Then for $p, p', p'' \in M$, there is a mapping

$$\alpha_1: J^r(M)(p, p') \times J^r(M)(p, p') \to J^r(M)(p, p') \tag{1.4}$$

Consider $\phi_1, \phi_2 \in G^r(M)$ such that:

$$\phi_1(p) = p', \ \phi_2(p') = p''.$$

Then, define α_1 by the following relation:

$$\alpha_1(j^r(\phi_1)(p), j^r(\phi_2)(p')) = j^r(\phi_2\phi_1)(p). \tag{1.5}$$

Exercise. Show that there is a map α_1 defined by relation (1.5).

Similarly, define a map

$$\alpha_2: J^r(p, p') \to J^r(p', p) \tag{1.6}$$

so that:

$$\alpha_2(j^r(\phi_1)(p)) = j^r(\phi_1{}^{-1})(p')). \tag{1.7}$$

Clearly, these mappings α_1, α_2 defined on $J^r(G(M))$ reflect the group structure on $G(M)$.

Now we can turn to one of Lie's basic ideas, a mechanism for generating groups via differential equations:

<u>Definition</u>. Let M be a manifold: An <u>r-th order Lie structure on M</u> is defined by a submanifold $N \subset J^r(G(M))$ such that:

$$\alpha_1(J^r(p, p') \cap N \times J^r(p', p'') \cap N) \subset J^r(p, p'') \cap N \tag{1.8}$$

$$\alpha_2(J^r(p, p') \cap N) \subset J^r(p', p) \cap N. \tag{1.9}$$

Given such a Lie structure, set $G(N)$ = set of $\phi \in G(M)$ such that:

$$j^r(\phi)(p) \in N \text{ for all } p \in M. \tag{1.10}$$

<u>Exercise</u>. Show that $G(N)$ is a subgroup of $G(M)$.

<u>Remarks</u>. The groups $G(N)$ defined in this way, as the set of all solutions of the differential equations (1.10), are what were meant by "Lie groups" in the 19th century. The "Lie groups" as we know them were called "finite parameter continuous groups."

For simplicity, we have chosen to define $G(N)$ "globally." It is also important to allow the possibility of considering diffeomorphisms between open subsets of M. This leads to what C. Ehresmann calls a "Lie pseudo group," a concept which is important for deeper work in this area, but which we will ignore at the moment.

Now, the product group $G(M) \times G(M)$, acts as a transformation group on $J^r(G(M))$. First, consider the following action of: $G(M) \times G(M)$ on $G(M) \times M$.

$$(\phi_1, \phi_2)(\phi, p) = (\phi_1\phi\phi_2^{-1}, \phi_2 p) \qquad (1.11)$$

for $\phi_1, \phi_2 \in G(M)$, $(\phi, p) \in G(M) \times M$.

Exercise. Show that (1.11) defines $G(M) \times G(M)$ as a transformation group on $G(M) \times M$.

Exercise. Show that this transformation group action of $G(M) \times G(M)$ on $G(M) \times M$ passes to the quotient to define a transformation group action on $J^r(G(M))$. This action is called the r-th prolongation of $G(M) \times G(M)$.

It is then characterized by the following relation:

$$(\phi_1, \phi_2)(j^r(\phi)(p)) = j^r(\phi_1\phi\phi_2^{-1})(\phi_2 p) \qquad (1.12)$$

Exercise. Suppose that N is a submanifold of $J^r(G(M))$ that defines a Lie structure on M. Show that the above action of $G(N) \times G(N)$ on $J^r(G(M))$ maps N into itself.

Example. Cartan Groups

In his monumental study of the groups $G(N)$ associated with Lie structures, E. Cartan defined a special class of such groups, and showed that, in a certain sense, a general group was isomorphic to one of his type. We shall not attempt here to describe this aspect of Cartan's work, which is still rather murky, but shall merely give the definition of this special class of group, which we will naturally call by Cartan's name.

Let M be a manifold, with a set $(\omega_1, \ldots, \omega_n)$ of everywhere independent 1-forms given on M. Suppose also that the Pfaffian system defined by the $\omega_1, \ldots, \omega_n$ are completely integrable, i.e. such that $d\omega_1, \ldots, d\omega_n$ belong to the Grassman ideal generated by the $\omega_1, \ldots, \omega_n$.

Set r = 1. Let N' be the subset of $G(M) \times M$ consisting of the pairs (ϕ, p) such that:

$$\begin{aligned} \phi^*(\omega_1)(p) &= \omega_1(p) \\ &\vdots \\ \phi^*(\omega_n)(p) &= \omega_n(p). \end{aligned} \tag{1.13}$$

Exercise. If $(\phi, p) \in N'$, and if $(\phi', p') \in G(M) \times M$ is equivalent to (ϕ, p), show that $(\phi', p') \in N'$.

Let N be the subset of $J^1(G(M))$ resulting from projection N' defined above from $G(M) \times M$ into $J^1(G(M))$. In view of the last exercise, N can also be defined as the collection of 1-jets $j^1(\phi)(p)$ of $\phi \in G(M)$ which satisfy (1.13).

Exercise. Show that N is a Lie structure for M. Show that G(N) is the set of all $\phi \in G(M)$ such that:

$$\phi^*(\omega_1) = \omega_1, \ldots, \phi^*(\omega_n) = \omega_n. \tag{1.14}$$

This is the form in which Cartan defines the group.

Problem: Suppose that $n = \dim M$, and that:

$$d\omega_i = c_{jki}\omega_j \wedge \omega_k, \tag{1.15}$$

$$(1 \le i, j \le n),$$

where (c_{ijk}) are constants. Suppose in addition that the Riemannian metric

$$\omega_i \cdot \omega_i$$

is complete. Show that the group G(N) defined by (1.14) is a Lie group (in the modern sense), which has the following properties:

a) G(N) acts transitively on M.

b) G(N) acts simply on M, i.e. if $\phi \in G(N)$ leaves a point of M fixed, then ϕ is the identity transformation. (Hence, G(N) may be identified, as a space, with M itself.)

c) The Lie algebra of G(N) has a basis whose structure constants are the (c_{ijk}) defined by (1.15).

This problem shows that "Lie groups" in the modern sense may be defined as a special sort of Cartan group. One of the basic ideas in Cartan's theory of the "infinite dimensional" groups is that they are to be studied as a generalization of the "finite dimensional" ones, defined in this way by a set of forms $(\omega_1, \dots, \omega_n)$, with

$$n < \dim M.$$

<u>Example 1</u>. <u>Linear fractional transformations</u>

Suppose that $M = R$, with x denoting the coordinate of M. Consider the group of diffeomorphisms of M of the following form:

$$x \to \frac{ax+b}{cx+d} = y(x), \tag{1.16}$$

where a, b, c, d are real constants. Exhibit this group as the group $G(N)$, where N is a Lie structure on $J^3(G(M))$. (Of course, in this simple case, one should carry out the details explicitly, and find the equations for N, in terms of the differential equations satisfied by $y(x)$).

<u>Example 2</u>. <u>Distance preserving motions in the plane</u>

Set $M = R^2$. Consider the group of diffeomorphism $M \to M$ which preserve Euclidean distance. Write down the formulas for the group transformations in terms of "finite parameters" (in the form similar to (1.16)), and again find differential equations satisfied by the transformations of the group. In terms of those differential equations, write down the submanifold N of $J^1(G(M))$ which defines the group.

2. THE LIE ALGEBRA OF A LIE STRUCTURE

Return to the general situation, M, G(M), G(M) × M, $J^r(G(M))$ are as before with G(M) × G(M) acting as a transformation group on $J^r(G(M))$. Let "1" denote the identity element of G(M). Then, G(M) × (1) is an invariant subgroup of G(M) × G(M). Consider a one parameter subgroup $t \to (\phi_t, 1)$ of G(M) × (1). It has an infinitesimal generator, acting on M, that we denote by X. X is a vector field on M:

$$X(f) = \frac{\partial}{\partial t} \phi_{-t}{}^*(f)/_{t=0} \tag{2.1}$$

Now, the group $t \to (\phi_t, 1)$ acting on $J^r(G(M))$ has an infinitesimal generator vector field on that we denote by X^r, which is an element of $V(J^r(G(M)))$, that we call the <u>prolongation of X</u>. One can show that the mapping

$$X \to X^r$$

is a Lie algebra homomorphism of V(M) into $V(J^r(G(M)))$ and that it is one-one.

Let N be a submanifold of $J^r(G(M))$ that defines Lie structure on M. Set:

$$\underset{\sim}{G}(N) = \text{set of vector fields } X \in V(M) \text{ such that } X^{r} \text{ is tangent to } N. \tag{2.2}$$

Then, $\underset{\sim}{G}(N)$ is a Lie algebra (under Jacobi bracket) of vector fields on M. It is called the Lie algebra of the Lie structure.

Problem: Suppose that $X \in V(M)$ generates a global one-parameter group of diffeomorphisms of M. Show that this group belongs to G(N) if and only if X belongs to $\underset{\sim}{G}(N)$ defined above.

Remark. This property of $\underset{\sim}{G}(N)$ is that which justifies calling it the "Lie algebra" of G(N).

One might also formulate a "localization" of this definition. This would lead to the concept of a "sheaf of Lie algebras of vector fields." However, we shall not elaborate on such a possible formulation.

This way of defining a class of Lie algebras of vector fields may be generalized to connect up with D. C. Spencer's theory of "infinite dimensional Lie groups" (See Kumpera and Spencer [1]). We shall briefly give the relevant definition:

Definition. Let M be a manifold, and let Γ be an $F(M)$-module. Then, a linear differential operator $D\colon V(M) \to \Gamma$ is called a Lie differential operator if the following condition is satisfied:

$$\begin{aligned}&\text{For } X, Y \in V(M), \text{ if } D(X) = D(Y) = 0,\\ &\text{then } D([X, Y]) = 0.\end{aligned} \tag{2.3}$$

In other words, the kernel of D is a Lie algebra of vector fields on M, which we denote by $\mathcal{G}(D)$. For example, for the Cartan groups defined in the previous section it is obvious how to define such a differential operator, so that $\mathcal{G}(D) = \mathcal{G}(N)$. Namely, set $\Gamma = F(M) \oplus \dots \oplus F(M)$

$$D(X) = X(\omega_1) \oplus \dots \oplus X(\omega_n). \tag{2.4}$$

Spencer's theory proceeds by the study of complexes of differential operators, with D the initial operator.

3. PROLONGATION OF LIE STRUCTURES

Let M and M' be manifolds, with $\sigma\colon M \to M'$ an onto, maximal-rank mapping. Suppose that the fibers $\sigma^{-1}(p')$ of σ are connected. For $p \in N$, let $M_p^r(\sigma)$ be the space of r-tangent vectors v to M at point p such that:

$$\sigma_*(v) = 0.$$

Geometrically, $M_p{}^r(\sigma)$ consists of the tangent vectors at p that are tangent to the fibers of σ through p.

Definition. A diffeomorphism φ: $M \to M$ is said to be r-projectible at p if:

$$\varphi_*(M_p{}^r(\sigma)) = M_{\varphi(p)}{}^r(\sigma). \tag{3.1}$$

ϕ is said to be globally projectible under σ if there is a diffeomorphism ϕ': $M' \to M'$ such that:

$$\phi'\pi = \pi\phi. \tag{3.2}$$

Exercise. Show that ϕ is projectible under σ if and only if $j^r(\phi)(p)$ is projectible under p, for each $p \in M$.

Now define $J^r(G(M), \sigma)$ as the projection in $J^r(G(M)$ of the subset of $(\phi, p) \in G(M) \times M$, such that ϕ is projectible under σ, at p.

Definition. Let N be a Lie structure for M. Then, N is said to be projectible under σ if:

$$N \subset J^r(G(M), \sigma). \tag{3.3}$$

Let us examine the geometric meaning of condition (3.3). Let G(N) be the group of diffeomorphisms of M defined by N. Thus, for $\phi \in G(N)$, $j^r(\phi)(M) \subset N \subset J^r(G(M), \sigma)$, hence, because of the above exercise, there is a $\phi' \in G(M')$ satisfying (3.2). The mapping $\phi \to \phi'$ then defines a homomorphism of $G(N)$ onto a group of diffeomorphisms of M', that we denote by $G(N, M')$. We say that $G(N)$ is a prolongation of $G(N, M')$.

Thus, a prolongation is a "homomorphism," in the abstract-group theoretical sense, that is geometrically realized in this way. Such a prolongation notion is a key element in Cartan's theory; his major goal is to classify groups up to prolongation. One can also formulate an analogous notion for "prolongation" of Lie algebras of vector fields defined by linear differential operators.

After this brief excursion into the realm of generalities, let us turn to the study of more specific topics.

4. ORDINARY DIFFERENTIAL EQUATIONS WHICH ARE LIE SYSTEMS IN THE RESTRICTED SENSE

One finds buried in the work of Lie and his disciples (particularly Cartan and Vessiot) many ideas for the study of the structure of the set of all solutions of certain special classes of differential equations. Typically, these special sorts of differential equations will be those that have a relation to a given group of transformations on a manifold. In the next sections we shall study various examples of systems of this sort. For simplicity, we remain within the class of ordinary differential equations, although the ideas are rather general and probably extend in some form to systems of partial differential equations.

In this section, we shall review certain ideas related to systems which we shall call "Lie systems in the restricted sense." We shall discuss the definition of these systems in modern language, and discuss some of their properties.

Let M be a manifold. Suppose that $t \to X_t$ is a one parameter family of vector fields on M, i.e. a curve in V(M).

<u>Definition</u>. A curve $t \to \sigma(t)$, $a \le t \le b$ in M is an <u>integral curve</u> of the vector field family $\{X_t\}$ if:

$$\sigma'(t) = X_t(\sigma(t)) \tag{4.1}$$

for $a \le t \le b$.

Let us look at the meaning of equation (4.1) in local coordinates (x_i), $1 \le i, j \le m = \dim M$, for M. Suppose that:

$$X_t = A_i(x, t) \frac{\partial}{\partial x_i}.$$

Then, if $x_i(t) = x_i(\sigma(t))$, condition (4.1) means that

$$\frac{dx_i}{dt} = A_i(x(t), t). \tag{4.2}$$

This is a system of ordinary differential equations for the $x_i(t)$. It is of the "non-autonomous" type, because the right hand side of (4.2) depends explicitly on t. The standard existence - uniqueness theorems for such systems apply.

Another way of looking at (4.1) is to add a variable to convert the problem into one of finding integral curves of a vector field on a space of one higher dimension. Set:

$$M' = R \times M, \text{ with } t \text{ the variable on } R.$$

$$X = \frac{\partial}{\partial t} + X_t. \tag{4.3}$$

Exercise. Show that a curve $t \to \sigma(t)$ in M is an integral curve of the family $\{X_t\}$ if and only if its graph curve $t \to (t, \sigma(t))$ is an integral curve of the vector field X given by (4.3).

Definition. Let $\underset{\sim}{G}$ be a Lie algebra of vector fields on a manifold M, and let $\{X_t\}$ be a one parameter family of vector fields on M. Then, the differential equations for the integral curves of $\{X_t\}$ are said to form a Lie system of the restricted type relative to the Lie algebra $\underset{\sim}{G}$ if, for each t, the vector field X_t belongs to $\underset{\sim}{G}$.

Let us now investigate the relation between the Lie system itself and G. First, let us give more definitions.

Definition. A one parameter family $t \to \phi_t$ of diffeomorphisms of a manifold M is called a flow on M. The curves $t \to \phi_t(p)$, with $p \in M$ held fixed, are called the orbits of the flow.

Definition. Let $\{X_t\}$ be a family of vector fields on M. Then, a flow $\{\phi_t\}$ is said to be generated by $\{X_t\}$ if each orbit of the flow is an integral curve of the $\{X_t\}$. Conversely, $\{X_t\}$ is called the infinitesimal generator of the flow $\{\phi_t\}$.

The following result gives the precise relation between the flow $\{\phi_t\}$ and its infinitesimal generator $\{X_t\}$.

Theorem 4.1. Let $\{\phi_t\}$ be a flow on M. Then, a family $\{X_t\}$ of vector fields is its infinitesimal generator if and only if:

$$X_t(f) = \phi_t^{-1*} \frac{\partial}{\partial t} \phi_t^*(f) \tag{4.4}$$

for all $f \in F(M)$.

Proof. Suppose that $\sigma(t) = \phi_t(p)$ is an integral curve of $\{X_t\}$. Then, by definition,

$$\sigma'(t) = X_t(\sigma(t)).$$

Hence, for $f \in F(M)$,

$$\sigma'(t)(f) = \frac{d}{dt} f(\sigma(t))$$
$$= X_t(f)(\sigma(t)),$$

or

$$\frac{d}{dt} f(\phi_t(p)) = X_t(f)(\phi_t(p)),$$

or

$$\frac{d}{dt} \phi_t{}^*(f) = \phi_t{}^*(X_t(f)). \tag{4.5}$$

This proves (4.4). The converse follows, because the steps are reversible.

Definition. Let G be a group of diffeomorphisms of a manifold M. A map $t \to \phi_t \in G$, $-\infty < t < \infty$, i.e. of $R \to G$, is called a one-parameter subgroup of G, if the following conditions are satisfied:

a) $\phi_{t+s} = \phi_t \phi_s$ for $t, s \in R$.

b) The map $R \times M \to M$, which sends (t, p) into $\phi_t(p)$, is C^∞.

If $t \to \phi_t$ is such a one-parameter subgroup, a vector field $X \in V(M)$ is its infinitesimal generator if:

$$X(f) = -\frac{\partial}{\partial t} \phi_t{}^*(f)_{t=0}$$

for $f \in F(M)$.

<u>Definition</u>. Let $\mathcal{G}$ be a Lie algebra of vector fields on M and let G be a group of diffeomorphisms of M. Then, $\mathcal{G}$ is said to be the <u>Lie algebra</u> of G if the following conditions are satisfied:

a) Each one-parameter subgroup of G has an infinitesimal generator which belongs to $\mathcal{G}$.

b) This mapping (one-parameter subgroups) $\to \mathcal{G}$ is onto.

<u>Definition</u>. Let $\{X_t\}$ be a family of vector fields on a manifold M. Let G be a group of diffeomorphisms of M. Then $\{X_t\}$ is said to be a <u>Lie system (of the restricted type) relative to G</u> if there is at least one flow $\{\phi_t\}$ on M whose infinitesimal generator is $\{X_t\}$, such that $\phi_t \in G$ for all t.

This definition suggests that we investigate the properties of different flows generated by the same vector field system:

<u>Problem</u>. Let $\{\phi_t\}$ and $\{\phi_t'\}$ be two flows on M. Show that they have the same infinitesimal generator if and only if there is a diffeomorphism ϕ: M $\to$ M such that:

$$\phi_t = \phi\phi_t'. \tag{4.6}$$

<u>Remark</u>. This result reflects a basic observation, that the equations (4.4) relating the flow and its infinitesimal

generator are invariant under left-translation of flows: $\{\phi_t\} \to \{\phi\phi_t\}$. Another way of looking at this is to say that the correspondence (flow) $\to$ (infinitesimal generator) is one-one provided we normalize the flows so that they are the identity map at one value of t, e.g. $t = 0$.

Problem. Let G be a "Lie group" in the form it is thought of nowadays, i.e. G is both a (finite dimensional) manifold and a group, with the group operations C^∞. Let G act on M in a C^∞ way. Let $\mathfrak{G}$ be its Lie algebra, identified with the collection of one parameter subgroups of G, hence also - with the transformation group action on M - with a Lie algebra of vector fields on M. Let $\{X_t\}$ be a one-parameter family of vector fields on M. Show that it defines a Lie system relative to $\mathfrak{G}$ if and only if it forms a Lie system relative to G, i.e. that the "local" and "global" notions of a "Lie system" coincide.

The most interesting sort of Lie systems for the applications involve groups which are "infinite dimensional." For example, hydrodynamics deals with Lie systems relative to the group of volume preserving diffeomorphisms; Hamiltonian mechanics deals with Lie systems relative to the group of automorphisms of a symplectic structure.

Let us now pause in the development of the general theory of Lie systems in order to discuss some instructive examples.

5. EXAMPLES OF LIE SYSTEMS OF THE RESTRICTED TYPE

<u>Example 1</u>. <u>Riccati equations</u>

Let $M = R$, with the coordinate on R denoted by "x". Let $\mathcal{G}$ be the Lie algebra of vector fields on M of the following form:

$$X = (a + bx + cx^2) \frac{\partial}{\partial x}, \tag{5.1}$$

with a, b, c, real constants.

Then, a Lie system relative to $\mathcal{G}$ is a family $\{X_t\}$ of vector fields on M of the form:

$$X_t = (a(t) + b(t)x + c(t)x^2) \frac{\partial}{\partial x}. \tag{5.2}$$

The integral curves of such a system are the curves $t \to x(t)$ in M such that:

$$\frac{dx}{dt} = X_t(x(t)). \tag{5.3}$$

Combining (5.2) and (5.3), we have:

$$\frac{dx}{dt} = a(t) + b(t)x + c(t)x^2. \tag{5.4}$$

The differential equation (5.4) is what is known, classically, as a "Riccatti equation."

Now, one knows that the group G generated by $\underset{\sim}{G}$ is the group of linear functional transformations: $x \to \frac{\alpha x+\beta}{\gamma x+\delta}$ of R. (See DGCV, p. 44. Of course, this is not quite exact from the global point of view. One must "compactify" R by adding a "point at infinity." However, the situation is so simple here that we shall keep to the classical way of thinking.)

<u>Exercise</u>. Let G be the group SL(2, R) of (2 × 2) real matrices of determinant one. Given the system (5.4), show that there is a curve

$$t \to \begin{pmatrix} \alpha(t), & \beta(t) \\ \gamma(t), & \delta(t) \end{pmatrix}$$

in G such that the solution $t \to x(t)$ of (5.4) is given by the following formula:

$$x(t) = \frac{\alpha(t)x(0) + \beta(t)}{\gamma(t)x(0) + \delta(t)}. \tag{5.5}$$

Interpret this result in terms of the general ideas described in Section 4.

<u>Example 2</u>. <u>Linear systems of ordinary differential equations</u>

Now, let $M = R^n$. Choose indices and the summation convention as follows:

$1 \le i, j \le n$.

Consider a family of vector fields $\{X_t\}$ on M of the following form:

$$X_t = a_{ij}(t)x_j \frac{\partial}{\partial x_i}. \tag{5.6}$$

The integral curves of this system are of the following form:

$$\frac{dx_i}{dt} = a_{ij}(t)x_j \tag{5.7}$$

or, introducing vector-matrix notation $x = (x_i)$, $a = (a_{ij})$,

$$\frac{dx}{dt} = a(t)x. \tag{5.8}$$

Let $\underset{\sim}{G}$ be the Lie algebra of vector fields on M spanned by the vector fields on M of the form (5.6), with constant coefficients (a_{ij}). Then, $\underset{\sim}{G}$ is isomorphic to the Lie algebra of $G = GL(n, R)$, the group of $n \times n$ real matrices with non-zero determinant. We see that $\{X_t\}$ defines a Lie system relative to $\underset{\sim}{G}$. The global action of G is the "natural" action of GL(n, R) on R^n, and we see that the system is also a Lie system relative to G.

Example 3. n-th order ordinary linear differential equations

Consider such an equation first in its classical form:

$$\frac{d^n x}{dt^n} + a_1(t) \frac{d^{n-1}x}{dt^{n-1}} + \dots + a_n(t)x = 0, \tag{5.9}$$

with $a_1(t), \dots, a_n(t)$ given functions.

There is a classical trick to reduce this to the previous case. Set:

$$x_1 = x.$$

$$\frac{dx_1}{dt} = x_2, \ldots, \frac{dx_{n-1}}{dt} = x_n. \tag{5.10}$$

Then,

$$\frac{dx_n}{dt} = -a_1(t)x_n - \ldots - a_n x_1. \tag{5.11}$$

Let us now construct the following vector field on R^n:

$$X = x_2 \frac{\partial}{\partial x_1} + \ldots + x_n \frac{\partial}{\partial x_{n-1}}$$
$$(a_1(t)x_n + \ldots + a_n x_n) \frac{\partial}{\partial x_n}. \tag{5.12}$$

This is a linear vector field on R^n, i.e. it generates a curve in $GL(n, R)$, and the projection $R^n \to R$ of $(x_1, \ldots, x_n) \to x_1$ maps the set of all of its integral curves onto the space of all solutions of (5.9). What has been essentially done is to "prolong" the system (5.9) to a Lie system on R^n.

Example 4. Hamilton equations as Lie systems associated to the group of symplectic automorphisms

Suppose $M = R^{2n}$. Adopt coordinates (x_i, y_i) for M, $1 \le i, j \le n$. Let $h(x, y, t)$ be a function on $M \times R$. Consider the following differential equations:

$$\frac{dx_i}{dt} = \frac{\partial h}{\partial y_i}\,(x(t)_1 y(t)_1 t) \tag{5.13}$$

$$\frac{dy_i}{dt} = -\,\frac{\partial h}{\partial x_i}\,(x(t)_1 y(t)_1 t).$$

For fixed t, set:

$$X_t = \frac{\partial h}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}\frac{\partial}{\partial y_i}. \tag{5.14}$$

Then, one sees immediately that the solutions of (5.13) are the integral curves of the vector field system $\{X_t\}$. Set:

$$\omega = dy_i \wedge dx_i. \tag{5.15}$$

Let G be the group of diffeomorphisms $\phi\colon M \to M$ such that:

$$\phi^*(\omega) = \omega.$$

Such a diffeomorphism is called a <u>symplectic automorphism</u>. Its Lie algebra is then contained in the Lie algebra of vector fields $X \in V(M)$ such that:

$$X(\omega) = 0. \tag{5.16}$$

<u>Exercise</u>. If X_t is defined by (5.14), show that it satisfies (5.16).

<u>Exercise</u>. If X_t generates a global flow on M, (ϕ_t), such that: ϕ_0 = identity, show that:

$$\phi_t^*(\omega) = \omega \text{ for all } \omega.$$

In other words, show that (X^t) is a Lie system with respect to G. Show how this flow may be used to write down the solutions of (5.18).

6. SYMMETRIES IN THE BROAD SENSE OF SYSTEMS OF DIFFERENTIAL EQUATIONS

So far in this chapter we have been discussing special situations and concepts which originated in Lie's work on the interconnection between group theory and the theory of differential equations. Now we shall broaden our outlook in order to define very general ideas, particularly aiming towards an explanation of what might be called "Lie systems in the broad sense." (What Vessiot called "systèmes automorphes.")

We shall now utilize the very general way of describing differential equations presented in GPS. Let M be a manifold. Let n and r be non-negative integers with:

$$n \le \dim M.$$

Let $C^r(M, n)$ be the manifold of r-th order contact elements of submanifolds of dimension n of M.

Definition. An r-th order differential equation for M, denoted typically by E, is a submanifold of $C^r(M, n)$. A submanifold map $\phi: N \to M$ (with dim N = n) is a solution of E if

$$j^r(\phi)(N) \subset E \tag{6.1}$$

S(E) denotes the space of solutions.

Let I be the "contact ideal" of forms on $C^r(M, n)$, i.e. the ideal generated by 1-forms (denoted in GPS by θ_a, $\theta_{a\mu}$, $\theta_{a\mu\nu}$, ...) such that the jets of n-dimensional submanifolds are integral manifolds of I.

If E is a differential equation, let I_E denote the ideal I restricted to the submanifold E. Then, if $\phi \in S(E)$, i.e. if ϕ is a submanifold map: $N \to M$ satisfying (6.1), then $j^r(\phi)$ is an n dimensional integral submanifold of I_E. However, there may be certain "exceptional" n-dimensional integral submanifolds of I_E which are not r-jets of submanifold maps. Accordingly, let us denote by S'(E) the set of all submanifold maps $\phi: N \to E$ which are integral submanifolds of I_E. (Goursat [1] calls these the "generalized solutions of the differential equation defined by E;" however, it would be somewhat confusing to use this terminology, since "generalized solution" now usually means "solutions in the distribution sense.") We shall call the elements of S'(E) the solutions in the broad sense of E.

Definition. Let E, E' be two submanifolds of $C^r(M, n)$, and let α: $C^r(M, n) \to C^r(M, n)$ be a diffeomorphism. Then the differential equations E, E' are said to be equivalent with respect to α if the following conditions are satisfied:

$$\begin{aligned} &\alpha(E) = E' \\ &\alpha^*(I_{E'}) = I_{E'}. \end{aligned} \tag{6.2}$$

Then, it follows from the definitions that:

$$\alpha(S'(E)) = S'(E'),$$

i.e. α sets up a correspondence between the solutions in the broad sense.

Now, let us formulate analogous ideas for families of differential equations.

Definition. Let $\mathfrak{E}$ be a family of differential equations for M, and let G be a group of diffeomorphisms of $C^r(M, n)$. Then, $\mathfrak{E}$ is said to be a Lie family of differential equations (relative to the group G) if the following condition is satisfied:

> For each $E \in \mathfrak{E}$, $\alpha \in G$, there is an $E' \in \mathfrak{E}$ such that (6.2) is satisfied, i.e. α is an equivalence between E and E'. (6.3)

$\mathcal{E}$ is said to be a <u>transitive Lie family</u> (relative to the group G) if it satisfies (6.3), and further, given E, E' $\in \mathcal{E}$, there is an $\alpha \in G$ which satisfies (6.2).

$\mathcal{E}$ is said to be a <u>Lie system</u> ϕ, <u>in the broad sense, relative to the group G</u>, if it is a transitive Lie family, and if the following further conditions are to be satisfied:

> Given E, E' $\in \mathcal{E}$, $\phi \in S'(E)$, $\phi' \in S'(E')$, where $\phi: N \to C^r(M, n)$, $\phi': N' \to C^r(M, n)$, are submanifold maps there is an $\alpha \in G$ which satisfies (6.2), and such that
> $$\alpha(\phi(N)) = \phi'(N'). \tag{6.4}$$

<u>Remark</u>: For simplicity, we have stated these have definitions of the Lie theory in this "global" form. Lie and his followers thought of these concepts "locally," which is, in fact, better from the point of view of rigorous mathematics because the space of <u>global</u> solutions of systems of differential equations is not an object about which, if the systems are at all complicated and/or nonlinear, very much is known. However, I believe it is easier for the reader if I do not go into all the definitions that would be necessary to talk about the "localization" of these notions. Further, note that (6.4) means that the submanifold $\phi(N)$ is transformed

into $\phi'(N')$, with perhaps a change of parameterization involved.

Definition. Let G be a group of transformations on $C^r(M, n)$, and let $\underset{\sim}{E}$ be a Lie family of differential equations, relative to the group G. For $E \in \underset{\sim}{E}$, let G_E denote the subgroup of elements $\alpha \in G$ that map E onto itself. G_E is called the symmetry subgroup of G at E. It acts as a transformation group on S'(E), the space of solutions of E. This action on the space of solutions is the key feature in the "Picard-Vessiot theory," a Galois theory for linear ordinary differential equations.

With these general ideas in hand, let us return to the consideration of examples.

7. SOME CLASSICAL EXAMPLES OF LIE FAMILIES OF DIFFERENTIAL EQUATIONS

One interesting feature of the various types of "Lie families" of differential equations considered above is that they encompass - after suitable specialization - many of the systems which appear most often in physical and geometric applications. Very often, special features of of the solutions of these equations have an interesting "explanation" in terms of the action of the group G on the systems that go to make up the family.

Example 1. First order partial differential equations for one unknown function

Here, we are concerned with a single real-valued function f on $C^1(M, m-1)$, where M is an m-dimensional manifold. E is the subset defined by: $f = 0$. Let $\underset{\sim}{E}$ denote the family of subsets $C^1(M, m-1)$ defined in this way. Let I be the ideal of contact forms on M. Let G be the group of "contact transformations," i.e. the diffeomorphisms $\alpha: C^1(M, m-1) \to C^1(M, m-1)$ such that:

$$\alpha^*(I) = I.$$

Then, we see that $\underset{\sim}{E}$ is a Lie family relative to the group G.

The action of G on $\underset{\sim}{E}$ has very strong "transitivity" properties. In fact, from a local point of view - and ignoring the possible singular and/or non-transversal solutions (which, we may recall, Goursat calls "generalized solutions") the action on G is that which was called in Section 6 a "Lie system in the broad sense," i.e. G acts transitively" on the solutions. The details needed to prove this are discussed very extensively in the classical literature, and we shall not pursue them here. (See the works of Goursat, Cartan and Caratheadory.) The basic technical tool in this work is the "canonical form" for closed two-forms on manifolds, and the standard results on their integral submanifolds. (For example, see DGCV, Chapter 13, for a brief treatment of these matters.) One interesting qualitative feature of this situation is that one can usually operate quite adequately within the domain of C^∞ mappings. However, the deeper study of the singularity structures - which has not even been attempted yet - will probably require real-analyticity assumptions.

Example 2. Linear ordinary differential equations

Suppose $M = R^2$. We consider 1-dimensional submanifolds of the form $t \to (t, x(t))$. Consider $C^n(M, 1)$. It may be parameterized by coordinates $(t, x, y_1, \ldots, y_n)$. Thus, the n-jet of the curve $t \to (t, x(t))$ is the following curve.

$$t \to (t, x(t), \frac{dx}{dt}, \ldots, \frac{d^n x}{dt^n}).$$

To define a system E, let $(a_0(t), \ldots, a_{n-1}(t))$ be given functions of t. Let E denote the set of points $(t, x, y_1, \ldots, y_n)$ such that:

$$y_n + a_{n-1}(t)y_{n+1} + \ldots + a_0(t)x = 0. \tag{7.1}$$

Thus, S(E) consists of the functions $t \to x(t)$ such that:

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_0 x = 0, \tag{7.2}$$

i.e. the space of all solutions (in the classical sense) of an n-th order, homogeneous ordinary differential equation with variable coefficients. These objects were studied extensively and brilliantly in the 19th century; the highlights are associated with such names as Gauss, Sturm, Liouville, Riemann, Fuchs, Poincaré, Picard and Vessiot.

Consider the following contact 1-forms on $C^n(M, 1)$:

$$\begin{aligned} \omega_1 &= dx - y_1\, dt \\ \omega_2 &= dy_1 - y_2 dt \\ &\vdots \\ \omega_n &= dy_{n-1} - y_n dt. \end{aligned}$$

Let I be the Grassman ideal they generate. Let I_E denote this ideal restricted to E.

Set:

$$\begin{aligned} &f(t, x, y_1, \ldots, y_{n-1}) \\ &\qquad = -a_{n-1}(t) y_{n-1} - \ldots - a_0(t) x. \end{aligned} \tag{7.3}$$

Then, E has coordinates $(t, x, y_1, \ldots, y_{n-1})$. I_E is generated by the following forms:

$$\begin{aligned} \omega_1 &= dx - y_1\, dt \\ \omega_2 &= dy_1 - y_2 dt \\ &\vdots \\ \omega_n &= dy_n - f\, dt \end{aligned} \tag{7.4}$$

We shall now let G denote the group of diffeomorphisms $\alpha: C^n(M, 1) \to C^n(M, 1)$ which are linear in the variables (x, y) (with variable coefficients, however), of the following form:

$$\begin{aligned} \alpha^*(x) &= \alpha_{00}(t)x + \alpha_{01}(t)y_1 + \dots + \alpha_{0,n-1}(t)y_{n-1} \\ \alpha^*(y_1) &= \alpha_{10}(t)x + \dots + \alpha_{1,n-1}(t)y_{n-1} \\ &\vdots \\ \alpha^*(y_{n-1}) &= \alpha_{n-1}(t)x + \dots + \alpha_{n-1,n-1}y_{n-1} \\ \alpha^*(y_n) &= \alpha_n(t)x + \dots + y_n. \end{aligned} \tag{7.5}$$

Thus, α maps the subset E defined by (7.1) into a similar system E', defined by functions $b_0(t), \dots, b_{n-1}(t)$. Then, the family of linear systems $\underset{\sim}{E} = \{E\}$ forms a Lie family of Differential Equations, relative to the group G.

Suppose now we are given two systems E, E' $\in \underset{\sim}{E}$ defined by two sets of functions $(a_0(t), \dots, a_{n-1}(t))$, $(b_0(t), \dots, b_{n-1}(t))$. We shall now investigate directly the conditions α given by (7.5) must satisfy in order that it map system E into system E'. For simplicity, we shall only consider the case $n = 2$. Then, we have, explicitly:

$$f = - a_0 x - a_1 y_1$$

$$f' = -b_0 x - b_1 y_1$$

$$\omega_1 = dx - y_1\ dt$$

$$\omega_2 = dy_1 - f\ dt$$

$$\omega_2' = dy_1 - f'\ dt$$

$$\alpha^*(x) = \alpha_{00}(t)x + \alpha_{01}(t)y_1$$

$$\alpha^*(y_1) = \alpha_{10}(t)x + \alpha_{11}(t)y_1$$

$$\alpha^*(y_2) = \alpha_{20}(t)x + \alpha_{21}(t)y_1 + y_2$$

$$\alpha^*(y_2 - f) = y_2 - f' \tag{7.6}$$

Thus, we have:

$$\alpha^*(f) - f' = \alpha_{20}x + \alpha_{21}y_1. \tag{7.7}$$

This shows that α_{20}, α_{21} are determined by $\alpha^*(x)$, $\alpha^*(y_1)$, i.e. by the action of α on the first order jets.

Now, the condition that

$$\alpha^*(I_E) = I_{E'},$$

is the existence of functions ρ_{11}, ρ_{12}, ρ_{21}, ρ_{22} such that:

$$\alpha^*(\omega_1) = \rho_{11}\omega_1 + \rho_{12}\omega_2' \tag{7.8}$$

$$\alpha^*(\omega_2) = \rho_{21}\omega_1 + \rho_{22}\omega_2'. \tag{7.9}$$

<u>Exercise</u>. Show that (7.8) is equivalent to the following conditions:

$$\begin{aligned} &\alpha_{00} = \rho_{11}, \ \alpha_{01} = \rho_{12} \\ &\frac{d}{dt}(\alpha_{00})x + \frac{d}{dt}(d_{01})y_1 \\ &\qquad = \alpha_{10}x + \alpha_{11}y_1 - \alpha_{00}y_1 - \alpha_{01}f'. \end{aligned} \tag{7.10}$$

Show that (7.9) is equivalent to the following conditions:

$$\begin{aligned} &\rho_{21} = \alpha_{10}; \ \alpha_{11} = \rho_{22} \\ &\frac{d}{dt}(\alpha_{10})x + \frac{d}{dt}(\alpha_{11})y_1 \\ &\qquad = \alpha^*(f) - \alpha_{10}y_1 - \alpha_{11}f'. \end{aligned} \tag{7.11}$$

Thus, we conclude from (7.10)-(7.11) that α is determined by an ordinary, linear differential equation for the matrix

$$C(t) = \begin{pmatrix} \alpha_{00}(t), & \alpha_{01}(t) \\ \alpha_{10}(t), & \alpha_{11}(t) \end{pmatrix}. \tag{7.12}$$

We can now reformulate these results in a form that is more standard in differential equation theory. Introduce the following vectors and matrices:

$$v(t) = \begin{pmatrix} x(t) \\ y_1(t) \end{pmatrix};$$

$$A(t) = \begin{pmatrix} 0, & 1 \\ -a_0(t), & -a_1(t) \end{pmatrix} \qquad (7.13)$$

$$B(t) = \begin{pmatrix} 0, & 1 \\ -b_0(t), & -a_1(t) \end{pmatrix}$$

Exercise. Show that $\alpha(E') = E$, $\alpha^*(I_E) = I_{E'}$, if and only if the transformation

$$v(t) \to C(t)v(t) = u(t) \qquad (7.14)$$

transforms the vector differential equation:

$$\frac{dv}{dt} = A(t)v$$

into the equation:

$$\frac{du}{dt} = B(t)u.$$

Further, show that the necessary and sufficient condition that the transformation (7.14) in fact transforms one system into the other is that:

$$\frac{dc}{dt} = BC - CA. \qquad (7.15)$$

In particular, we may consider the case where $B = A$, i.e. the system E = the system E'. Then, we see from (7.15) that the map $C \to C(0)$ maps G_E into $GL(2, R)$, the group of 2×2, invertible, real matrices.

Remarks. These results, which are essentially trivial, of course, are still extremely important as a link between Lie's theory of groups and differential equations and the Picard-Vessiot theory.

If A = B, the set of all solutions of (7.15) forms a group under multiplication. Thus, this differential equation is the defining equation for the symmetry group of the differential equation.

Chapter II

DISSIPATIVE MECHANICAL SYSTEMS AND ELECTRICAL NETWORKS

1. INTRODUCTION

In GPS I have presented a general differential-geometric way of describing classical mechanical systems, of the type, for example, described in Whittaker's book on mechanics [1]. In this chapter, we will investigate briefly systems that might be called "dissipative." We shall, in fact, follow an idea due to Lord Rayleigh, and study such systems via a "dissipative potential." Then, we shall proceed to study electrical networks as examples of such systems. Later Chapters and Part B will deal with electric circuits in more detail, with particular attention to their differential geometric structure and possible generalizations.

2. THE RAYLEIGH DISSIPATIVE POTENTIAL

Let N be a configuration space manifold, of coordinates (q_a), $1 \le a, b \le n$. Let $M = R$, with t the variable on M, and $E = N \times R$. Let $J^1(E)$ be the bundle of 1-jets of cross-sections. Then, (q_a, t) are coordinates for E, while $(q_a, \dot{q}_a, t)$ are coordinates for $J^1(E)$. Consider a system defined by a Lagrangian $L(q, \dot{q}, t)$ and a force form $\omega = F_a(q, \dot{q}, t)(dq_a - \dot{q}_a\, dt)$. Recall then from GPS that Newton's laws can be written in the following form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\left(q(t), \frac{dq}{dt}, t\right)\right) - \frac{\partial L}{\partial q_a}\left(q(x), \frac{dq}{dt}, t\right) = F_a\left(q(t), \frac{dq}{dt}, t\right). \tag{2.1}$$

<u>Definition</u>. A function $\mathcal{R}(q, \dot{q}, t)$ on $J^1(E)$ is said to be a <u>dissipative potential</u> for ω if:

$$F_a = \frac{\partial \mathcal{R}}{\partial \dot{q}_a}. \tag{2.2}$$

This is the classical definition, due to Lord Rayleigh. (See Whittaker, [1], p. 231). Let us see how it may be put into coordinate-free terms. Notice that L, $\mathcal{R}$ are both functions on $J^1(E)$. One may define their Cartan forms in the usual way:

$$\theta(L) = L dt + \frac{\partial L}{\partial \dot{q}_a}(dq_a - \dot{q}_a\, dt). \tag{2.3}$$

$$\theta(\mathcal{R}) = R dt + \frac{\partial \mathcal{R}}{\partial \dot{q}_a}(dq_a - \dot{q}_a\, dt). \tag{2.4}$$

Combining (2.1), (2.2)-(2.4), we see that the differential equations (2.1) take the following form:

$$\sigma^{1}{}'(t) \rfloor d\theta(L) = \theta(\mathcal{R}) - \mathcal{R}dt. \tag{2.5}$$

(Here, $t \to \sigma(t)$ is the curve $t \to (t, q(t))$ in $E = R \times N$, $t \to (t, q(t), \frac{dq}{dt}) = \sigma^1(t)$ is its one-jet.)

In particular, (2.5) puts the Rayleigh form of Lagrange's equations (i.e. (2.1)-(2.2)) into a coordinate-free form. Another way of putting this may be described as follows:

Theorem 2.1. With $\theta(L)$, $\theta(\mathcal{R})$ defined as one-forms on $J^1(E)$ by (2.3)-(2.4), let X be a vector field on $J^1(E)$ which satisfies the following condition:

$$X \rfloor d\theta(L) = \theta(\mathcal{R}) - \mathcal{R}dt. \tag{2.6}$$

Then, the integral curves of X are the solutions of the Lagrange-Rayleigh equations (2.1)-(2.2).

Let us now turn to the study of the "dissipative" properties of $\mathcal{R}$. Suppose that L is time-independent, i.e.

$$\frac{\partial}{\partial t}(L) = 0. \tag{2.7}$$

Thus, we know that also:

$$\frac{\partial}{\partial t}(\theta(L)) = 0. \tag{2.8}$$

(The left hand side of (2.8) denotes the Lie derivative of the 1-form $\theta(L)$ by the vector field $\frac{\partial}{\partial t}$). Set:

$$h = -\theta(L)(\tfrac{\partial}{\partial t}).$$

The function h is called the Hamiltonian or energy function of the Lagrangian L.

Exercise. Show that

$$dh = \frac{\partial}{\partial t} \lrcorner\, d\theta(L). \tag{2.9}$$

Deduce the "conservation of energy" from this formula, i.e. that $X(h) = 0$ if $\underset{\sim}{R} = 0$.

This property of h suggests the following definition.

Definition. The function $X(h)$ is called the power function of the system defined by L and $\underset{\sim}{R}$. (Physically, it is the rate of dissipation of energy).

Remarks: First, notice that $X(h)$ is independent of the choice of X satisfying (2.6). For, two such X's differ by a characteristic vector field of $d\theta(L)$, which the above exercise says will annihilate h. Second, this definition of "power" may be made when the right hand side of (2.6) is an arbitrary 1-form.

Let us now compute $X(h)$, using (2.9):

$$\begin{aligned} X(h) = dh(X) &= X \lrcorner \frac{\partial}{\partial t} \lrcorner\, d\theta(L) \\ &= -\frac{\partial}{\partial t} \lrcorner\, (X \lrcorner\, d\theta(L)) \\ &= -\frac{\partial}{\partial t} \lrcorner\, (\theta(\underset{\sim}{R}) - \underset{\sim}{R}\, dt) \end{aligned}$$

$$- \frac{\partial}{\partial t} \lrcorner \left(\frac{\partial \underset{\sim}{R}}{\partial \dot{q}_a} (dq_a - \dot{q}_a\, dt) \right)$$
$$= \frac{\partial \underset{\sim}{R}}{\partial \dot{q}_a} \dot{q}_a .$$

Let us now state this result as a separate theorem.

Theorem 2.2. If a Rayleigh potential $\underset{\sim}{R}$ exists for the force term in a dynamical system, then the power function is $\frac{\partial \underset{\sim}{R}}{\partial \dot{q}_a} \dot{q}_a$. In particular, if $\underset{\sim}{R}$ is homogeneous of degree 2 in the $\dot{q}_a$ variables (which is a common situation), then the power function is 2R.

These general properties of $\underset{\sim}{R}$ are the justification of thinking of it as a "dissipation function."

3. LAGRANGE-RAYLEIGH EQUATIONS THAT GENERATE GRADIENT FLOWS

Brayton and Moser [1] and Smale [1] have shown that the differential equations of electrical circuits may be written in such a form that they are the gradient curves of a function on a manifold with a Riemannian structure which is not positive. This feature is very interesting for two reasons - first, it is always important to know that a given class of physical systems may be "modelled" in a relatively simple and unified way, and, second, there is hope that the mathematical properties of the metric may be used in a non-trivial way in studying the physical system.

These facts suggest that it would be useful to study general mechanical systems that generate gradient flows. In this section I present some brief comments.

Continue the notations of Section 1: N has coordinates (q_a), $1 \le a, b \le n$, E has coordinates (q_a, t), $J^1(E)$ has coordinates $(q_a, \dot{q}_a, t)$. Let us suppose now that L and R are functions of the following form:

$$L = L(q, t), \text{ i.e. } L \text{ is a function on } N \times R \text{ of the "configuration space" variables } (q_a, t) \text{ alone.} \tag{3.1}$$

$$\mathcal{R} = \frac{1}{2} \beta_{ab}(q)\dot{q}_a\dot{q}_b. \tag{3.2}$$

Also, suppose that the problem is "non-degenerate," in the sense that

$$\text{determinant } (\beta_{ab}) \ne 0. \tag{3.3}$$

Further, we may without any loss in generality assume that $\beta_{ab} = \beta_{ba}$. With these choices, the Lagrange-Rayleigh equations, (2.1)-(2.2), take the following form:

$$-\frac{\partial L}{\partial q_a}(q(t), t) = \beta_{ab}(q(t))\frac{dq_b}{dt}. \tag{3.4}$$

We shall now show how these equations may be interpreted geometrically, in terms of gradient curves of a function on a Riemannian manifold. Let β be the F(N)-bilinear, symmetric map: $V(N) \times V(N) \to F(N)$ defined as follows:

$$\beta(\frac{\partial}{\partial q_a}, \frac{\partial}{\partial q_b}) = \beta_{ab}. \qquad (3.4)$$

Using (3.4), we see that β defines a Riemannian metric on N (which is not necessarily a positive metric, of course.)

Let $f \in F(N)$. Recall, from DGCV, pg. 277, that grad f is the vector field on N such that:

$$\beta(\text{grad } f, X) = X(f) \text{ for } X \in V(N). \qquad (3.5)$$

<u>Definition</u>. Suppose that $t \to f^t$ is a one-parameter family of functions on N. Thus, for each t, grad f^t is a vector field on N. We say that a curve $t \to \sigma(t)$ in N is a <u>gradient curve</u> of f^t if:

$$\sigma'(t) = \text{grad } f^t(\sigma(t)) \text{ for all } t. \qquad (3.6)$$

Let us now see how equations (3.4) may be interpreted in terms of gradients.

Set:

$$f^t = - L(\ , t),$$

i.e. f^t is the negative of L, with t held fixed. Using (3.5), we have:

$$\beta(\text{grad } f^t, \frac{\partial}{\partial q_a}) = - \frac{\partial L}{\partial q_a},$$

hence:

$$\text{grad } f^t = - \beta_{ab}^{-1} \frac{\partial L}{\partial q_a} \frac{\partial}{\partial q_a} \tag{3.7}$$

(β_{ab}^{-1} denotes the inverse matrix to β_{ab}). Thus, (3.6) takes the following form, in these coordinates:

$$\frac{dq_a}{dt} = - \beta_{ab}^{-1} \frac{\partial L}{\partial q_b} (q(t), t).$$

We see that these are precisely the Lagrange-Rayleigh equations, (3.4), with the postulated choice for L and R.

We can now sum up in a coordinate-free way as follows.

<u>Theorem 3.1</u>. Let N be a manifold, E = N × R the product fiber space. Identify $J^1(E)$, the space of 1-jets of its cross-sections, with the product T(N) × R of the tangent bundle to N and the real line R. Let β be a tensor field on N, defined as a non-degenerate, bilinear symmetric map: T(N) × T(N) → R, defining a Riemannian metric for N. (Or, equivalently, as an F(N)-bilinear map V(N) × V(N) → F(N)).

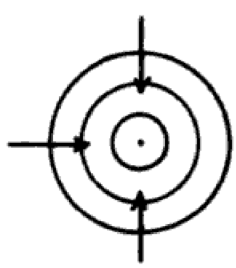
Let L be a function: N × R → R. Consider L as a function: $J^1(E)$ → R. Let $\underset{\sim}{R}$ be the function: T(N) → R defined as follows, using the metric β:

$$\underset{\sim}{R}(v) = \frac{1}{2} \beta(v, v). \tag{3.8}$$

Then the solutions of the Lagrange-Rayleigh differential equations (2.5), with "Lagrangian" L and "Rayleigh dissipative potential" $\mathcal{R}$, are precisely the curves $t \to \sigma(t)$ in N which are the gradient curves of the one parameter family of functions f^t on N defined by (3.6).

These facts have diverse ramifications: on the one hand, there has been considerable work in the mathematical literature concerning the global behavior of gradient curves, mainly in connection with the area of differential topology called "Morse Theory." (See Milnor [1]). Physically this result is of interest because it suggests a "variational" interpretation of the Lagrange-Rayleigh equations which is somewhat different, in spirit at least, from the "Hamilton principle" approach discussed in Chapter II. We shall now present a brief discussion of this point.

Suppose that $N = R^2$, the Euclidean plane, with the metric just the usual Euclidean metric. Let f be a function: $N \to R$. Then, the gradient lines of f with respect to the metric are the curves perpendicular to the level curves of f:



Thus, if the circles in the diagram represents the level curves of f, with the origin a maximum of f (or topographically, a "mountain peak") the arrows represent the gradient lines, which are of course the lines of "steepest ascent." As everyone knows from calculus, the gradients may be defined via a variational principle, which, intuitively, says that the tangent vector to the gradient curves point in the direction in which f increases "most rapidly."

Exercise. Formulate precisely a variational principle for the solutions of the Lagrange-Rayleigh equations, in the case we have been considering where they are identical with gradient curves for a metric on N defined by the dissipative potential. Compare this principle with the Hamilton principle, as discussed in GPS.

Now, there have been many attempts in the physics and engineering literature to define "variational principles" for "dissipative" physical systems. These all go back to such a principle for electric circuits invented by Maxwell which he called the "principle of minimal dissipation of heat." At the moment, we shall not pursue further the study of such principles; however, I hope to return to this in a systematic way in a later volume of this work.

We now turn to the study of electrical circuit equations, which provide interesting examples of physical systems of the type whose general properties we have been studying.

4. ABSTRACT ELECTRIC CIRCUIT THEORY

We shall present what a first sight looks like a rather abstract version of material that is standard in the electrical engineering literature as "circuit theory." In particular, we shall put much less emphasis on the theory of graphs than is customary. (For example, the best text-books on circuit theory from our point of view is that by Desoer and Kuh [1]). However, the approach presented here is better suited to studying certain mathematical features of the theory, and to certain generalizations we have in mind.

First, let us recall certain standard jargon of linear vector space theory.

Definition. Let A be a real finite dimensional vector space. Then, the dual space of A, denoted by A^d, is the space of all R-linear maps $a^d: A \to R$.

If B is a linear subspace of A, then the perpendicular subspace to B denoted by $B^{\perp}$, is the space of $a^d \in A^d$ such that:

$$a^d(B) = 0. \tag{4.1}$$

Exercise. Show that:

$$\dim B + \dim B^{\perp} = \dim A = \dim A^d.$$

(Here, "dim" means the dimension of these spaces as real vector spaces).

Definition. An ordered pair (A, I) consisting of a real finite dimensional vector space A and a linear subspace $I \subset A$ is called a system of currents. The elements of I, usually denoted by "i", are called currents.

Definition. Let (A, I) be such a system of currents. Let A^d denote the dual vector space to A, and let $V = I^{\perp} \subset A^d$, i.e. V consists of the vectors $v \in A^d$ such that:

$$v(I) = 0. \tag{4.2}$$

Then, the pair (A^d, V) is called a system of voltage associated with the system of currents (A, I). The elements of V, usually denoted by "v", are called voltages.

From now on in this Section, suppose such space (A, I), (A^d, V) are given. Then, (4.1) takes the following form:

$$\dim I + \dim V = \dim A = \dim A^d. \tag{4.3}$$

The aim of "electrical circuit theory" is to define certain curves $t \to (i(t), v(t))$ in $I \times V$ that may be identified with the time varying "currents" and "voltages" in actual electrical circuits. To do this, we must postulate a further mathematical structure on A, which corresponds to the decomposition of elements in an electrical circuit into "resistors," "inductors" and "capacitors."

<u>Definition</u>. An <u>RLC-decomposition</u> of A is a decomposition of A as the direct sum of three linear subspaces, denoted by A_R, A_L, A_C. In the notation of linear algebra, we have then:

$$A = A_R \oplus A_L \oplus A_C. \tag{4.4}$$

We shall need a few concepts from linear algebra, which we shall briefly review.

<u>Definition</u>. Let X be a vector space, and let Y be a linear subspace of X. Let X^d be the dual space of X. Then, a linear subspace $Z \subset X^d$ is said to be <u>transversal to Y</u> if the map $Z \to Y^d$ which, to each $z \in Z \subset X^d$, assigns the linear functional $y \to z(y)$ on Y, is an isomorphism between Z and Y^d.

<u>Exercise</u>. If X is finite dimensional, show that $Z \subset Y^d$ is

transversal to Y, in the sense of the above definition, if and only if X^d is a direct sum of $Y^\perp$ and Z.

The question arises of how to construct such a transversal subspace. The following result gives such a method: Return to the decomposition (4.4) of A. Set:

$$B = A^d. \tag{4.5}$$

$$B_R = (A_L + A_C)^\perp. \tag{4.6}$$

$$B_L = (A_R + A_C)^\perp. \tag{4.7}$$

$$B_C = (A_R + A_L)^\perp. \tag{4.8}$$

Exercise. Show that

$$B = B_R \oplus B_L \oplus B_C. \tag{4.9}$$

Show also that B_R is transversal to A_R; that B_L is transversal to A_L; and that B_C is transversal to A_C. Show also that:

$$B_R = A_L^\perp \cap A_C^\perp. \tag{4.10}$$

$$B_L = A_R^\perp \cap A_C^\perp. \tag{4.11}$$

$$B_C = A_R^\perp \cap A_L^\perp. \tag{4.12}$$

Now, set:

$$n = \dim A = \dim B$$

$$n_R = \dim A_R = \dim B_R$$

$$n_L = \dim A_L = \dim B_L$$

$$n_C = \dim A_C = \dim B_C.$$

Definition. Suppose given the current-voltage system (A, I), $(B = A^d, V = I^\perp)$. Suppose also given RLC decompositions (A_R, A_L, A_C), (B_R, B_L, B_C) of A and B given by (4.4) and (4.6)-(4.8). Then, a set of constitutive relations for the system is a triple (N_R, N_L, N_C) of manifolds satisfying the following conditions:

N_R is a submanifold of $A_R \times B_R \times (-\infty, \infty)$. (4.13)

N_L is a submanifold of $A_L \times B_L \times (-\infty, \infty)$. (4.14)

N_C is a submanifold of $A_C \times B_C \times (-\infty, \infty)$. (4.15)

$\dim N_R = n_R (\equiv \dim A_R)$. (4.16)

$\dim N_L = n_L$. (4.17)

$\dim N_C = n_C$. (4.18)

N_R, N_L, N_C are called, respectively, the resistor, inductor and capacitor constitutive manifolds.

Definition. An abstract electrical circuit system is given by the following data:

A current system (A, I)

A voltage system $(B = A^d, V = I^{\perp})$

A triple (N_R, N_L, N_C) of resistor-inductor-capacitor constitutive manifolds.

Suppose such an electrical circuit system is given. We are prepared to define the electrical circuit differential equations. Given a curve $t \to (i(t), v(t))$. $-\infty < t < \infty$, in $A \times B$, define curves:

$$t \to (i_R(t), v_R(t)) \in A_R \times B_R$$

$$t \to (i_L(t), v_L(t)) \in A_L \times B_L$$

$$t \to (i_C(t), v_C(t)) \in A_C \times B_C,$$

so that

$$i(t) = i_R(t) + i_L(t) + i_C(t). \tag{4.19}$$

$$v(t) = v_R(t) = v_L(t) + v_C(t), \tag{4.20}$$

i.e. (i_R, i_L, i_C), (v_R, v_L, v_C) are the projections of i, v in the subspaces defined by the RLC decompositions (4.4) and (4.9).

<u>Definition</u>. A curve $t \to (i(t), v(t))$, $-\infty < t < \infty$, in $A \times B$ is called a <u>solution curve</u> of the system of electrical circuit systems if the following conditions are satisfied:

$$i(t) \in I \text{ for all } t \tag{4.21}$$

$$v(t) \in V \text{ for all } t \tag{4.22}$$

$$(i_R(t), v_R(t)) \in N_R \text{ for all } t \tag{4.23}$$

$$(\tfrac{d}{dt} i_L(t), v_L(t)) \in N_L \text{ for all } t \tag{4.24}$$

$$(i_L(t), \tfrac{d}{dt} v_L(t)) \in N_C \text{ for all } t. \tag{4.25}$$

<u>Remarks</u>:

a) (4.21) and (4.22) are the abstract versions of what are usually called <u>Kirchoffs current and voltage laws</u>.

b) The choice of $(-\infty, \infty)$ is not crucial, of course. For certain purposes $0 \le t < \infty$ would be more appropriate.

Chapter III

PFAFFIAN, VECTOR FIELD AND MONGE SYSTEMS

1. VECTOR FIELD AND PFAFFIAN SYSTEMS

We shall first give the definition of these objects. Let M be a manifold, which is usually fixed throughout the discussion. Recall that F(M) denotes the commutative associative algebra (over the real numbers) of C^∞, real-valued functions on M, and that V(M) denotes the C^∞ vector fields on M.

V(M) is both a Lie algebra, with the real numbers as field of scalars, and an F(M)-module. The ideas discussed in this section involve the study of the "interaction" between these two structures.

$F^1(M)$ denotes the one-differential forms on M. It is defined as the dual F(M)-module to V(M). Thus, $\theta \in F^1(M)$ is defined as an F(M)-linear map

$$\theta: V(M) \to F(M).$$

The exterior derivative, d, maps

$$F^1(M) \to F^2(M).$$

The Lie algebra operation, $(X, Y) \to [X, Y]$, is called the Jacobi bracket, and plays a basic role in the study of the differential-geometric properties of M. The exterior derivative and Jacobi bracket are related via the following formula:

$$d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]) \tag{1.1}$$

for X, $Y \in V(M)$.

Thus, in a sense the exterior derivative and Jacobi bracket are "dual" to each other. Usually, geometric properties expressed in terms of one can also be translated in terms of the other. For example, Lie preferred to use vector fields and the Jacobi bracket as the basic objects, while Cartan used differential forms and the exterior derivative. We shall see that there is a similiar duality in the theory of differential equations, between "vector field" and "Pfaffian" systems."

Definition. A vector field system on M is an F(M)-submodule of V(M). A Pfaffian system on M is an F(M)-submodule of $F^1(M)$.

Let $V \subset V(M)$ denote a vector field system. One can define a Pfaffian system P(V) as follows:

$$P(V) = \{\theta \in F^1(M): \theta(V) = 0\}. \tag{1.2}$$

P(V) is called the dual system to V. Similiarly if

$P \subset F^1(M)$ is a Pfaffian system, set:

$$V(P) = \{X \in V(M): \theta(X) = 0\} \tag{1.3}$$

for all $\theta \in P$.

$V(P)$ is called the <u>dual system to P</u>.

Thus, the assignment

$$V \to P(V) \subset F^1(M)$$

$$P \to V(P) \subset V(M)$$

sets up a reciprocity in the study of these geometric structures.

Suppose given a vector field system V and a Pfaffian system P on a manifold M. We shall now define the geometric notion of an "integral curve" of both. We shall see in Section 2 and 3 that many of the ideas of systems theory can be conveniently and elegantly described in terms of the "integral curve" notion.

<u>Definition</u>. Let $t \to \sigma(t)$, $a < t < b$, be a curve in M. Let P be a Pfaffian system, and let V be a vector field system, on M. Then, σ is an <u>integral curve of P</u> if:

$$\theta(\sigma'(t)) = 0 \tag{1.4}$$

for all $\theta \in P$, $a < t < b$.

($\sigma'(t)$ denotes the tangent vector to σ at t, an element of $M_{\sigma(t)}$).

σ is an <u>integral curve of V</u> if the following condition is satisfied:

$$\begin{array}{l} \text{For each real number } c,\ a < c < b, \text{ there} \\ \text{is a small real number } \varepsilon > 0 \text{ and a vector} \\ \text{field } Z \in V \text{ such that} \\ \qquad \sigma'(t) = Z(\sigma(t)) \\ \qquad \text{for } c - \varepsilon < t < c + \varepsilon. \end{array} \tag{1.5}$$

<u>Remarks</u>: Notice that these definitions of "integral curves" have been phrased in such a way that they apply to broader classes of curves than just C^∞ ones. For example, they apply to continuous, piecewise C^∞ curves, as indicated by the following diagram:

Relation (1.5) says that such a piecewise C^∞ curve is an integral curve of V if and only if it is composed of pieces, each of which is an integral curve of a single vector field which belongs to V.

<u>Exercises</u>

a) Modify the definitions so that they apply to piecewise C^1 curves.

b) If σ is an integral curve of V, show that it is an integral curve of the dual Pfaffian system $P = P(V)$.

c) Let us say that the vector field system V is <u>free</u> if there is an $F(M)$-basis $Z_1, \ldots, Z_m$ for $V(M)$ such that $Z_1, \ldots, Z_n$ is an $F(M)$-basis for V. If V is free in this sense, show that a curve σ is an integral curve of V if and only if it is an integral curve of dual $P(V)$.

d) Construct a vector field system V so that $P(V)$ has integral curves which are not integral curves of V. In other words, the "duality" operation $V \to P(V)$ may increase the supply of integral curves. This is a sort of "pathology" vector field systems may have, related to its "geometric" singularities.

We shall now turn to a class of examples of Pfaffian and vector field systems that is very important in control and systems theory.

2. PFAFFIAN AND VECTOR FIELD SYSTEMS DEFINED BY INPUT-STATE RELATIONS

As basic introductory references for control and systems theory, we shall use the books by Brockett and Desoer [1,1] and Volume VIII.

Let X and U be manifolds, of dimension M and N respectively. Choose indices as follows:

$$1 \le i, j \le n$$
$$1 \le a, b \le m.$$

Suppose that (x_i) is a coordinate system of functions on X and that (u_a) is a coordinate system of functions on U. Thus, a point of X is denoted by x, while a point of U is denoted by u. We shall write:

$$x = (x_i)$$
$$u = (u_a),$$

meaning (with a slight ambiguity that the reader must keep straight for himself) that $(x_1, \ldots, x_n)$ are the coordinates for the point x, and that $(u_1, \ldots, u_m)$ are the coordinates of u. X is called the <u>state space</u>, while U is called the <u>input space</u>.

Consider a set of ordinary differential equations of the following form:

$$\frac{dx_i}{dt} = f_i(x(t), u(t), t). \tag{2.1}$$

Given a curve $t \to u(t)$, $0 \le t < \infty$, in input space, and a point x^0 in state space, the curve $t \to x(t)$, $u(t)$, $0 \le t < \infty$, in (state space) $\times$ (input space) is determined as the solution of (2.1). Thus, we have a mapping:

$$\text{(curves in U)} \times X \to \text{(curves in } X \times U)$$

Consider the manifold:

$$M = X \times U \times R.$$

Consider (x_i, u_a, t) as functions on M. They also form a coordinate system for M. We shall define a Pfaffian and vector field system on M whose integral curves are essentially the curves defined by (2.1).

Set:

$$\theta_i = dx_i - f_i\, dt. \tag{2.2}$$

These are Pfaffian forms on M. (A "Pfaffian form" is a more picturesque name for a 1-differential form, i.e. an element of $F^1(M)$). Let P be the F(M)-submodule of $F^1(M)$ generated by the forms θ_i.

<u>Exercise</u>. Show that P is a free Pfaffian system, i.e. that the elements $\theta_1, \ldots, \theta_n$ are a basis for P as an F(M)-module, and that there is an F(M)-basis for V(M) whose first M elements are the $\theta_1, \ldots, \theta_m$.

Let us now examine the relation between the integral curves of P and the solutions of the "input-state" differential equations (2.1).

<u>Definition</u>. Let σ: R $\to$ M be a curve in M. σ is said to

be transversal to the function t if $\sigma^*(dt)$, as a one-form on R, is non-zero at each point of R.

Consider such a transversal curve. Suppose "s" denotes the parameter on R, thus,

$$\sigma^*(dt) = h(s)ds,$$

where $s \to h(s)$ is a mapping $R \to R$. "Transversality" means that

$$h(s) \neq 0 \text{ for all } s \in R.$$

Hence,

$$h(s) > 0$$

or

$$h(s) < 0$$

for all $s \in R$.
Thus, we can change the parameterization of the curve (possibly changing the orientation as well, of course) so that:

$$\sigma^*(t) = t.$$

Exercise. Prove this statement.

Hence, the curves in M which are transversal to t can, after at most a change in orientation and parameterization, be written in the form:

$$t \to (x(t), u(t), t).$$

where $t \to x(t)$ is a curve in the "state space" X, and $t \to u(t)$ is a curve in the "input space" U.

Now we are prepared to prove the following result, which of course is trivial once the notation is understood:

Theorem 2.1. A curve in M is an integral curve of P and transversal to the "time" function t if and only if it can be reoriented and reparameterized so that it is of the form

$$t \to (x(t), u(t), t),$$

where $t \to (x(t), u(t))$ is a solution of (2.1).

Of course, the structure of the integral curves of P which are not transversal to t must also be considered. One class consists of the curves $\sigma: R \to M$ for which:

$$\sigma^*(dt) = 0. \tag{2.3}$$

By (2.2), we must also have:

$$\sigma^*(dx_i) = 0.$$

Hence,

$$d\sigma^*(t) = d(\sigma^*(x_i)) = 0,$$

i.e. t and x_i are constant along σ. Thus, they are the curves of the form:

$$s \to (t^0,\ x^0,\ u(s)),$$

where t^0, x^0 are fixed. Physically, they may be considered as curves on which the input "jumps."

The curves which neither satisfy (2.3) nor are transversal represent certain "singular" situations, whose study we shall pass by for the moment.

These results are essentially trivial, but do provide an important "geometric" setting for the ideas of systems theory. They also open the way for the study of the standard ideas of systems theory (such as "controlability" and "observability") by differential-geometric methods, particularly by the case of the Caratheodory-Chow accessibility theorem. (See Chow [1], Hermann [2,4,5], DGCV, Jurdjevich and Sussman [1] and Sussman [1].)

There is an alternate description in terms of a vector field system on M. Let V be the F(M)-submodule of V(M) spanned by the following vector fields:

$$\begin{aligned} &Z = \frac{\partial}{\partial t} + f_i \frac{\partial}{\partial x_i} \\ &\frac{\partial}{\partial u_a}. \end{aligned} \tag{2.4}$$

Exercise. Show that V is the dual vector field system to P, and that V is a free vector field system.

Exercise. Show directly that a curve in M is an integral curve of V if and only if it is an integral curve of P.

3. INPUT-STATE-OUTPUT RELATIONS IN TERMS OF MANIFOLD THEORY

Suppose that X, U, $M = X \times U \times R$ have the same meaning as in Section 2. Introduce another manifold Y, called the output space. Suppose

$$\dim Y = p.$$

Introduce indices as follows:

$$1 \leq \alpha, \beta \leq p.$$

Suppose (y_α) is a coordinate system on Y. Denote a point of Y by

$$y = (y_\alpha).$$

Let $f_i(x, u, t)$, $g_a(x, u, t)$ be given functions of the indicated variables, i.e. real-valued functions on M. We will say that they define a system.

Definition. A curve $t \to (u(t), y(t))$ in $U \times Y$ is said to be an input-output curve for the system if there is a curve $t \to x(t)$ in X such that:

$$\begin{aligned} \frac{dx_i}{dt} &= f_i(x(t), u(t), t) \\ y_\alpha(t) &= g_\alpha(x(t), u(t), t). \end{aligned} \tag{3.1}$$

Now, "system theory" may be regarded as the study of such input-output curves. (See Zadeh and Desoer [1]). Alternately, one may call the whole set-up symbolized by the equations (3.1) a "dynamical system," and regard it as a generalization of the "dynamical system" notion which first was abstracted from the mathematics of classical and quantum mechanics.

However, there is also a useful "geometric" setting for (3.1). Set:

$$N = Y \times X \times U \times R. \tag{3.2}$$

Consider

$$y_\alpha - g_\alpha(x, u, t) = z \tag{3.3}$$

as functions on N. Set:

$$N' = \{\text{points of } N \text{ on which } z_\varkappa = 0\}.$$

$$\theta_i = dx_i - f_i\, dt \tag{3.4}$$

$$P' = \text{Pfaffian system on } N' \text{ generated by the } \theta_i. \tag{3.5}$$

Notice that N' is a submanifold of N. Set:

$$\theta_i' = \theta_i \text{ restricted to } N'.$$

Let π_{in}, π_{out} be the maps

$$\pi_{in}: N' \to U$$

$$\pi_{out}: N' \to Y,$$

defined as follows:

$$\begin{aligned} \pi_{in}(y, x, u, t) &= u \\ \pi_{out}(y, x, u, t) &= y. \end{aligned} \tag{3.6}$$

Now, we have:

<u>Theorem 3.1</u>. A curve $t \to (u(t), y(t))$ in $U \times Y$ is an input-output pair of the system if and only if there is a curve $t \to \sigma(t)$ in N' such that:

$$\sigma^*(t) = t \tag{3.7}$$

$$\sigma \text{ is an integral curve of } P' \tag{3.8}$$

$$\pi_{in}(\sigma(t)) = u(t) \tag{3.9}$$

$$\pi_{out}(\sigma(t)) = y(t). \tag{3.10}$$

This result again shows how a standard problem of system theory may be interpreted geometrically. Again, although this interpretation is "trivial", it provides a setting whereby the Caratheodory-Chow theorem (and certain related differential-geometric ideas) may be used to study the standard "systems theory" problems, such as "controllability" and "observability."

4. THE DERIVED SYSTEMS OF A PFAFFIAN SYSTEM

We now return to "pure" differential geometry. Let M be a manifold, with P a Pfaffian system on M. Now, the set of differential forms of all degrees on M forms an algebra, with the "exterior product" $\wedge$ as the algebra multiplication. This is called the Grassman algebra of M. As for any algebra, a subset I of forms on M is called an ideal if it satisfies the following conditions:

$$\omega_1 + \omega_2 \in I$$

for $\omega_1, \omega_2 \in I$

$$\theta \wedge \omega \in I$$

for all $\omega \in I$, any form θ on M of arbitrary degree.

Since zero-degree forms are functions, and since exterior product by zero degree forms is just the "ordinary" product of functions, notice that a Grassman ideal of forms is also an F(M)-module.

$I(P)$ denotes the smallest Grassman ideal containing the set of forms P.

<u>Exercise</u>. Show that:

$$P = \{\theta \in I(P) \colon \theta \text{ is of degree } 1\}. \tag{4.1}$$

<u>Definition</u>. The <u>first derived Pfaffian system</u> of P, denoted by P^1, is defined as follows:

$$P^1 = \{\theta \in P \colon d\theta \in I(P)\}. \tag{4.2}$$

<u>Theorem 4.1</u>. P^1 is an $F(M)$-submodule of $F^1(M)$, i.e. P^1 defines a Pfaffian system on M.

<u>Proof</u>. It should be clear from (4.2) that P^1 is a linear subspace of P. We must show that it is invariant under multiplication by functions $f \in F(M)$.

But, for $\theta \in P^1$,

$$d(f\theta) = df \wedge \theta + fd\theta.$$

$df \wedge \theta \in I(P)$, since $\theta \in I(P)$ and $I(P)$ is a Grassman ideal, while, by hypothesis, $d\theta \in I(P)$, hence also $fd\theta \in I(P)$.

<u>Definition</u>. The <u>second derived system</u>, P^2, of P is the first derived system of P^1, i.e.

$$P^2 = (P^1)^1. \tag{4.3}$$

Similiarly, define the n-th derived system:

$$P^n = (P^{n-1})^1. \tag{4.4}$$

Thus, we have a descending sequence

$$P \supset P^1 \supset \ldots \tag{4.5}$$

of Pfaffian systems, called the <u>derived system sequence</u>.

<u>Exercise</u>. Suppose that P is free. Show that P is completely integrable, i.e. satisfies the hypotheses of the Frobenian complete integrability theorem, if and only if

$$P = P^1.$$

One may then regard the derived sequence (4.5) as geometric structure for the study of noncompletely integrable Pfaffian systems. In E. Cartan's Collected Works one will find many examples where this derived structure is studied, and correlated with important and interesting geometric properties.

Now, we turn to study the dual notion of "derived systems" for vector field systems. At first, we shall treat it independently, then show how the "duality" relates the two notions.

5. DERIVED VECTOR FIELD SYSTEMS

Let M be a manifold, and let $V \subset V(M)$ be a vector field system for M, i.e. an F(M)-submodule of V(M). Set:

V^1 = smallest F(M) submodule of
V(M) containing V and [V,V].

V^1 is called the first <u>derived system</u> of V.

Set:

$$V^2 = (V^1)^1$$

$$V^3 = (V^2)^1$$

and so forth. Then, we have:

$$V \subset V^1 \subset V^2 \subset \ldots \tag{5.1}$$

The sequence of vector field systems on M is called the <u>derived sequence defined by V</u>.

We shall now see how the derived system notion of Pfaffian systems, as defined in Section 4, and the derived system for vector fields, are related "by duality."

Let V be a given vector field system, and let $P \subset F^1(M)$ be the dual Pfaffian system, i.e.

$$P = \{\theta \in F^1(M)\colon \theta(V) = 0\}. \tag{5.2}$$

Set:

$$P^1 = \{\theta \in F^1(M)\colon d\theta \in I(P)\}. \tag{5.3}$$

(Recall from Section 4 that I(P) denotes the Grassman algebra ideal generated by I). Suppose first that:

$$\theta \in P^1, \; X \in V^1.$$

If $X \in V$, then

$$\theta(X) = 0.$$

If $X \in [\boldsymbol{V},\boldsymbol{V}]$, i.e. if X is a sum of elements of the form $[Y,Z]$, with $Y, Z \in V$, then

$$\theta([Y, Z]) = d\theta(Y,Z) - Y(\theta(Z)) - Z(\theta(Y)) = 0.$$

Hence, we have:

$$P^1 \subset P(V^1). \tag{5.4}$$

<u>Theorem 5.1</u>. If V is a free vector field system, then:

$$P^1 = P(V^1). \tag{5.5}$$

<u>Proof</u>. Suppose $X_1, \ldots, X_m$ is an F(M)-basis of V(M), such that $X_1, \ldots, X_n$ is a basis for V. Let $\theta_1, \theta_1, \ldots, \theta_m$ be the dual basis of $F^1(M)$, i.e.

$$\theta_i(X_j) = \delta_{ij} \tag{5.6}$$

for $1 \leq i, j \leq m$.

Suppose that

$$\theta \in P(V^1).$$

Now, there are functions f_{ij} such that

$$d\theta = \sum_{i,j=1}^{m} f_{ij}\theta_i \wedge \theta_j. \tag{5.7}$$

For $1 \le k, \ell \le n$, we have:

$$0 = \theta([X_k, X_\ell]) = - d\theta(X_k, X_\ell) = f_{k\ell}.$$

Hence, (5.7) takes the form:

$$\begin{aligned} d\theta &= \sum_{j=1}^{m} \sum_{i=n+1}^{m} \varphi_{ij}\theta_i \wedge \theta_j \\ &= - \sum_{i=n+1} \Big(\sum_{j=1} f_{ij}\theta_j \Big) \wedge \theta_i. \end{aligned}$$

But, $\theta_{n+1}, \ldots, \theta_m \in P(V)$. This proves that:

$$d\theta \in I(P),$$

hence that:

$$\theta \in P^1.$$

Together with (5.4), this proves (5.5).

As a Corollary of Theorem 5.1, we see that if all the derived systems

$$V \subset V^1 \subset V^2 \subset \ldots$$

are free, then

$$P^1 = P(V^1),\ P^2 = P(V^2), \ldots .$$

There is then a perfect "duality" between derived systems of vector fields and Pfaffians when they are free.

<u>Theorem 5.2</u>. Suppose that V, V' are free vector field systems on M such that:

$$V \subset V',$$

but

$$V \neq V'.$$

Then, the number of elements of an F(M)-basis of V is less than the number of elements in V'.

<u>Exercise</u>. Prove Theorem 5.2.

<u>Definition</u>. Let V be a vector field system on M. Then, V is said to have a <u>free derived system</u> if each of the vector field systems

$$V \subset V^1 \subset V^2 \subset \ldots$$

is free.

<u>Definition</u>. The vector field system V is said to be <u>completely integrable</u> if:

$$V = V^1. \tag{5.8}$$

<u>Theorem 5.3</u>. Let V be a vector field system on M whose derived system is free. Then, there is an integer r such that:

V^r is completely integrable,

but $V^{r-1} \neq V^r$.

This integer is called the <u>class</u> of the system.

Proof. Suppose r did not exist. Thus, we would have:

$$V \neq V^1 \neq V^2 \neq \ldots$$

But, by Theorem 5.2, the dimension of V^1, V^2,... as an F(M)-module must keep increasing, which would be a contradiction.

Remark: If r is the class of V, and if P is the dual Pfaffian system to V, then

$$P \neq P^1 \neq \ldots \neq P^r = P^{r+1}.$$

6. INTEGRAL FUNCTIONS OF VECTOR FIELD SYSTEMS

Let V be a vector field system on the manifold M. For simplicity, for the rest of this chapter we shall suppose that V and its derived systems are free. This means "geometrically" that the following "constant rank" condition is satisfied; for each integer r.

$$\dim V^r(p) \text{ is constant as } p \text{ ranges over } M. \quad (6.1)$$

Recall that

$$V^r(p) = \{X(p) \colon X \in V^r\}.$$

Note that $V^r(p)$ is a linear subspace of M_p. At a later point, we shall have to seriously face the possibility of "singularities", i.e. (6.1) not satisfied, but to understand the main geometric ideas it is reasonable to proceed with the "free" hypothesis.

Definition. A function $f \in F(M)$ is an integral of V if the following condition is satisfied:

$$X(f) = 0 \text{ for all } X \in V. \tag{6.2}$$

Theorem 6.1. f is an integral of V if and only if it is an integral of V^r, for all integers $r \geq 1$.

Proof. Since

$$V \subset V^r,$$

we see that an integral f of V^r is also an integral of V. Conversely, suppose f satisfies (6.2). Then, for X, Y $\in$ V,

$$\begin{aligned} 0 &= YX(f) - XY(f) \\ &= [Y,X](f). \end{aligned}$$

Hence,

$$Z(f) = 0 \text{ for all } Z \in V^1,$$

i.e. f is an integral of V^1. Iterating the argument, we see that f is an integral of $V^2 = (V^1)^1$, of $V^3 = (V^2)^1$, and so forth.

Theorem 6.2. Suppose that r is the smallest integer r such that V^r is completely integrable, i.e. such that $V^{r+1} = V^r$. Set:

$$s = \dim M - (\text{dimension of } V^r \text{ as an } F(M)\text{-module}).$$

Then, in a neighborhood of each point $p \in M$, there are s functionally independent integral functions $f_1,\ldots,f_s$ such that any other integral function is a function of them.

Proof. This is a consequence of the Frobenius Complete Integrability Theorem, Local Version. (See DGCV, p. 65, Theorem 8.3.)

Remark: The Caratheodory-Chow Accessibility Theorem (See DGCV, p. 249, Theorem 18.1) asserts that given $p \in M$, the leaf passing through p of the foliation defined by the completely integral system V^r(i.e. the maximal integral submanifolds of V^r) is the set of points that one can reach from p on piecewise C^∞, continuous integral curves of V. In turn, this theorem (and its refinements proved by the author [2, 4, 5] and by Jurdjevich and Sussman [1, 1]) is a basic result in optimal control theory.

Note a difference in terminology from that used in DGCV. There, "V" is denoted typically by "H", and V^r is denoted by D(H), and called the "derived system."

7. THE DERIVED SYSTEM OF A MONGE SYSTEM

Classically, Monge systems are systems of ordinary differential equations. (See DGCV, p. 254.) The differential equations of control-systems theory considered in

Sections 2 and 3 are examples of Monge systems. For the reader's convenience, we shall give the general definition again, in terms of manifold theory.

Definition. Let N be a manifold, and let

$$M = T(N) \times R \tag{7.1}$$

be the Cartesian product of its tangent bundle and the real numbers R. A Monge system on N is defined by giving a submanifold S of M. A curve $t \to \sigma(t)$ in N is called a solution or an integral curve of the Monge system if the following condition is satisfied:

$$\text{for each } t, \text{ the point } (\sigma'(t), t) \text{ lies in } S. \tag{7.2}$$

(Recall that $\sigma'(t) \in N_{\sigma(t)} \subset T(N)$ denotes the tangent vector to σ at t.) Of course, this can be formulated more generally in terms of jet-spaces (see GPS), but it will suffice for our purpose, at least for the moment, to think in this way.

We shall now see how Monge systems are defined in more classical terms, by means of local coordinates for N and M.

Suppose that

$$n = \dim N.$$

Choose indices and the summation convention on these indices as follows:

$$1 \le i, j \le n.$$

Let (x_i) be a coordinate system of functions on N. Define functions $(\dot{x}_i)$ on T(M) as follows:

$$\dot{x}_i(v) = dx_i(v) \tag{7.3}$$

for $v \in T(M)$.

Then,

$$(x_i\,,\ \dot{x}_j\,,\ t) \tag{7.4}$$

define a coordinate system for $M = T(N) \times R$.

Suppose that the submanifold S of M defining the Monge system is defined by setting real valued functions $f_1,\dots,f_m$ on M equal to zero. These functions then become functions of the coordinates (7.2). Choose indices as follows:

$$1 \le a\,,\ b \le m.$$

Then, S is defined by the following relations:

$$f_a(x\,,\ \dot{x}\,,\ t) = 0 \tag{7.5}$$

a solution of the Monge system defined by S is then a curve $t \to (x_i(t))$ which satisfies the following differential equations:

$$f_a(x(t),\ \frac{dx}{dt},\ t) = 0. \tag{7.6}$$

Let us describe these curves in terms of Pfaffian and vector field systems. Define one-forms on M as follows:

$$\theta_i = dx_i - \dot{x}_i\ dt. \tag{7.7}$$

Set:

$$\theta_i' = \theta_i \text{ restricted to } S. \tag{7.8}$$

P = Pfaffian system on S spanned by the θ_i'.

$V \subset V(S)$ dual vector field system to P.

Let us assume that V is a free vector field system.

Exercise. Determine the conditions that the functions (f_a) must satisfy in order that V be free.

The functions on M are what physicists call "Lagrangian," i.e. they appear as integrands in calculus of variations problems. A basic notion in the differential-geometric theory of the calculus of variation is a certain one-form associated with a Lagrangian function, called the Cartan form.((See DGCV for the theory of the Cartan form for "single integral" variational problems. The generalization to "multiple integral" problems is given in LAQM, VB, and GPS).

Definition. The Cartan form $\theta(f) \in F^1(M)$ associated with the function $f \in F(M)$ is defined by the following formula:

$$\theta(f) = f\,dt + \frac{\partial f}{\partial \dot{x}_i}\,\theta_i. \tag{7.9}$$

Set:

$$\theta(f)' = \theta(f) \text{ restricted to } S. \tag{7.10}$$

Theorem 7.1. The 1-forms $\theta(f_a)'$ on S defined by (7.9) and (7.10) belong to P^1, the first derived system associated with the Pfaffian system P defined by (7.8).

Proof. Let I(P) be the Grassman algebra ideal generated by the 1-forms in P. We must show that

$$d\theta(f_a)' \in I(P). \tag{7.11}$$

To prove (7.11), let us differentiate both side of (7.10):

$$d\theta(f_a) = da_a \wedge dt + \frac{\partial f_a}{\partial \dot{x}_i} \wedge d\theta_i + \ldots$$

(the terms ... indicate terms in I(P))

$$= df_a \wedge dt - \frac{\partial f_a}{\partial \dot{x}_i} d\dot{x}_i \wedge dt + \ldots$$

$$= (df_a - \frac{\partial f_a}{\partial \dot{x}_i} d\dot{x}_i) \wedge dt + \ldots$$

$$= (\frac{\partial f_a}{\partial x_i} dx_i) \wedge dt + \ldots$$

$$= (\frac{\partial f}{\partial x_i} (dx_i - \dot{x}, dt) \wedge dt + \ldots$$

(since $dt \wedge dt = 0$).

$$= \frac{\partial f}{\partial x_i} \theta_i \wedge dt + \ldots ,$$

which indeed belongs to I(P), and completes the proof of (7.11).

We shall now compute P^1 more precisely, in case where the Monge system is "regular," in the following sense.

Definition. The Monge system defined by the functions $f_a(x,\dot{x},t)$ is regular if the values of the one forms

$$df_a, \theta_i \tag{7.12}$$

are linearly independent at each point of S.

Exercise. Show that the Monge system is regular if and only if the rank of the matrix

$$(\frac{\partial f_a}{\partial \dot{x}_i})$$

is equal to m at each point of S.

Let us suppose that the Monge system is regular. We can then suppose also (after at most relabelling the coordinates) that the 1-forms

$$d\dot{x}_{m+1}, \ldots, d\dot{x}_n$$

are linearly independent when restricted to S. Choose indices and the corresponding summation convention as follows

$$m+1 \le u, v \le n.$$

Then, the following 1-forms, when restricted to S, form a basis of 1-forms:

$$\theta_i, d\dot{x}_u, dt.$$

From now on, we shall leave off the "prime" to indicate that the differential forms are restricted to S. Thus, P has a basis consisting of the θ_i. With this convention,

$$\theta(f_a) = \frac{\partial f_a}{\partial \dot{x}_i} \theta_i.$$

Suppose now that ω is a 1-form on S that belongs to the first derived system P^1. Then,

$$\omega = h_i \theta_i.$$

Again, use the notation ... to indicate terms that he with grassman ideal I(P) generated by P. Then,

$$\begin{aligned} d\omega &= h_i d\theta_i + \ldots \\ &= - h_i d\dot{x}_i \wedge dt + \ldots \end{aligned} \tag{7.13}$$

Suppose also that:

$$d\dot{x}_a = A_{au}d\dot{x}_u + A_a\,dt + \ldots \tag{7.14}$$

Then

$$d\omega = (-h_aA_{au} - h_u)d\dot{x}_u \wedge dt + \ldots$$

Notice that this can belong to I(P) if and only if

$$h_u = -\,h_aA_{au}. \tag{7.15}$$

In particular, this proves the following result:

Theorem 7.2. The derived system P^1 is of dimension m as an F(S)-module.

Now we can prove a main result:

Theorem 7.3. Suppose that the Monge system is regular. Then, the one-forms $\theta(f_a)$, when restricted to S, form an F(S)-basis for P^1.

Proof. We have:

$\theta(f_a) = \frac{\partial f_a}{\partial \dot{x}_i}\,\theta_i$ when restricted to S (since $f_a = 0$ on S). We know that the rank of the matrix $(\frac{\partial f_a}{\partial \dot{x}_i})$ is m. This implies that the forms $\theta(f_a)$ are linearly independent at each point of S. Combined with Theorems 7.1 and 7.2 this implies that the $\theta(f_a)$, when restricted to S, form an F(S)-basis of P^1, as required.

8. MONGE SYSTEMS DEFINED BY A CALCULUS OF VARIATIONS PROBLEM

In the classical calculus of variations, it is known that a "simple" variational problem can be converted to a

"Mayer" problem. We shall use this as motivation.

Introduce real variables (q_i), $1 \le i, j \le n$. They may be thought of as the "configuration space" variables of a classical mechanics problem. Also introduce the "velocity" variables $\dot{q}_i$, and the "time" variable t. Suppose given a function $L(q,\dot{q},t)$ of these variables called the Lagrangian. The "calculus of variations" problem is to find a curve $t \to q(t)$ in configuration space which extremizes the following integral:

$$\int L(q, \frac{dq}{dx_i}\ t)\ dt. \tag{8.1}$$

To treat this in the "Mayer" way, introduce another "configuration space" variable y, with its velocity $\dot{y}$. Let M be the space of variables $(q_i, \dot{q}_i, y, \dot{y}, t)$, and let f be the following function on M:

$$f = \dot{y} - L. \tag{8.2}$$

Let N be the space of variables (q_i, y). Then, f defines a "Monge system" on N. The solutions of the Monge system are the curves $t \to (q(t), y(t))$ such that:

$$\frac{dy}{dt} = L(q(t), \frac{dq}{dt}, t),$$

or

$$y = \int L(q(t), \frac{dq}{dt}, t)dt. \tag{8.3}$$

Our goal in this section is to see how the solutions $t \to q(t)$ of the Euler-Lagrange equations associated with the variational problem (8.1) are defined in terms of the

Pfaffian system P on the submanifold S of M defined by relations (8.2), generated by the following 1-forms

$$\theta_i = dq_i - \dot{q}_i\, dt$$

$$\theta = dy - \dot{y}\, dt.$$

By Theorem 7.3, we know that P^1, the first derived system of P, is generated by $\theta(f)$, where $\theta(f)$ is the Cartan form associated with f. Using 8.2, we see that:

$$\theta(f) = \theta(\dot{y}) - \theta(L).$$

Now,

$$\theta(\dot{y}) = \dot{y}\, dt + \frac{\partial \dot{y}}{\partial \dot{y}} (dy - \dot{y}\, dt) = dy.$$

Hence,

$$\theta(f) = dy - \theta(L). \tag{8.4}$$

This proves the following result:

Theorem 8.1. Let S be the submanifold of M defined by setting

$$\dot{y} = L.$$

Let P be the Pfaffian system generated by the 1-forms $dq_i - \dot{q}_i\, dt$ and $dy - \dot{y}\, dt$. Then, the derived system P^1 of P is generated by the 1-form $\theta(f)$, given by formula (8.4).

Suppose now that $t \to q(t)$, $0 \le t \le 1$, is a curve that is an extremal of the variational problem (8.1) defined by the Lagrangian L. In other words, suppose that the Euler-Lagrange equations are satisfied:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\left(q(t), \frac{dq}{dt}, t\right)\right) = \frac{\partial L}{\partial q_i}\left(q(t), \frac{dq}{dt}, t\right). \tag{8.5}$$

Consider the curve

$$t \to (q(t),\ \dot{q}(t) = \frac{dq}{dt},\ y(t),\ \dot{y}(t) = \frac{dy}{dt},\ t) = \sigma(t) \qquad (8.6)$$

in M, where:

$$y(t) = \int_0^t L(q(s), \frac{dq}{ds}, s)\ ds. \qquad (8.7)$$

Let $\sigma'(t)$ denote the tangent vector to this curve.

Let $\sigma_1(t)$ be the curve $t \to (q(t), \frac{dq}{dt}, t)$ in R^{2n+1}. We know (for example, from DGCV, Chapter 15), that $t \to \sigma_1(t)$ is a characteristic curve of the one-form $d\theta(L)$, i.e.

$$\sigma_1'(t) \rfloor d\theta(L) = 0. \qquad (8.6)$$

Using (8.4), we have:

$$d\theta(f) = -\ d\theta(L). \qquad (8.9)$$

We have now proved the main result of this section;

<u>Theorem 8.2</u>. Let M be the space of variables (q_i, y), and let N be the space of variables (q_i). Identify $T(M) \times R$ with the space of variables $(q_i, y, \dot{q}_i, \dot{y}, t)$. Let f be the real value function on $T(M) \times R$ defined by (8.2), and let π be the projection map: $T(M) \times R \to N$ which assigns q to the point $(q, y, \dot{q}, \dot{y}, t)$ of $T(M) \times R$.

Let S be the submanifold of $T(M) \times R$ defined by setting $f = 0$. Let P be the Pfaffian system on S generated by the following forms:

$$dq_i - \dot{q}_i\ dt,\ dy - \dot{y}\ dt. \qquad (8.10)$$

let $s \to \sigma(s)$ be a curve in $T(M) \times R$ which is transversal to the function "t", and which is an integral curve of P.

Then, the projection curve $s \to \pi\sigma(s)$ in N is an extremal of the variational problem (8.1) if and only if the curve $s \to \sigma(s)$ satisfies the following condition:

$$\sigma'(s) \lrcorner\, d\theta(f) = 0. \tag{8.11}$$

In turn, condition (8.11) is very important geometrically. It says that the wave $s \to \sigma(s)$ is a certain type of "singular curve" of the Pfaffian system P. We shall study this notion more systematically in the next section.

We can reformulate the conditions of Theorem 8.2 in terms of vector fields. Let V be the dual vector field system to P, i.e.

$$V = \{X \in V(S)\colon \theta(X) = 0 \text{ for all } \theta \in P\}.$$

Consider a vector field $X \in V$ which satisfies the following conditions:

$$X(t) = 1;\ X \lrcorner\, d\theta(f) = 0. \tag{8.12}$$

It is readily seen that the integral curves of X are when projected down to q-space, just the extremal curves (i.e. the solutions of the Euler-Lagrange equations) of the Lagrangian L.

Let V^1 be the first derived system of V. It is the smallest F(S)-submodule of V(S) containing V and [V,V]. It is also the dual system to P^1.

<u>Theorem 8.3</u>. Let $X \in V$. Then, $X \rfloor d\theta(f) = 0$ if and only if the following condition is satisfied:

$$[X, V^1] \subset V^1. \tag{8.13}$$

<u>Proof</u>. We know from the work of Section 7 that the 1-form $\theta(f)$ generates the dual Pfaffian system P^1. Suppose first that:

$$X \rfloor d\theta(f) = 0. \tag{8.14}$$

Then, for $Y \in V^1$,

$$\begin{aligned} 0 &= (X \rfloor d\theta(f))(Y) = d\theta(f)(X, Y) \\ &= X(\theta(f)(Y)) - Y(\theta(f)(X)) - \theta(f)([X, Y]) \\ &= -\theta(f)([X, Y]). \end{aligned} \tag{8.15}$$

Now, we know that:

$$V^1 = \{Z \in V(S): \theta(f)(Z) = 0\} \tag{8.16}$$

(8.15) and (8.16) combine to prove that

$$[X, Y] \in V^1, \tag{8.17}$$

which, since Y is arbitrary, proves (8.13).

Conversely, suppose that (8.13) is satisfied. We see that the steps leading to (8.17) are reversible, to prove (8.14).

Remark: In terms of the general terminology of the theory of Pfaffian and vector field systems, condition (8.13) means that X is a Cauchy characteristic vector field of V^1. In later work, we shall see how this is related to "accessibility" and "controllability" problems.

Chapter IV

THE CARATHEODORY APPROACH TO THE CALCULUS OF VARIATIONS

1. INTRODUCTION

I have already described one version of the Caratheodory approach in DGCV and my paper, "Some Differential-Geometric aspects of the Lagrange variational problems." In both of these references, it was assumed that the Lagrangian was "homogeneous," in the sense that its integrated value was independent of the parameterization of the curve. Now, this is not the most convenient form of the problem for discussing the applications to optimal control theory. Accordingly, in this chapter I will present an alternate treatment for time-dependent Lagrangians, in a form that is closer to that described by Caratheodory himself in his book [1].

However our development differs from Caratheodory's in that we mean to keep as close as possible to coordinate-free, geometric ideas developed in the theory of manifolds. In particular, I will use the "Cartan form" associated to a Lagrangian, whose properties are developed in DGCV (for simple-integral variational problems) and in LAQM, VB and GPS for multiple integral variational problems. For simplicity, we shall only deal with simple-integral problems in this chapter. I hope to treat multiple-integral problems in a similiar spirit at a later point in my work.

2. THE CARTAN FORM OF A LAGRANGIAN

Let N be a manifold. Recall that T(N) denotes its tangent bundle. Set:

$$M = T(N) \times R. \tag{2.1}$$

In terms of classical mechanics, N may be thought of as the "configuration" or "position" space of a mechanical system. M defined by (2.1) may be thought of as the space of position-velocity-time. It is the manifold on which the basic objects of the calculus of variations are defined.

Remark: M can be related to the "jet-bundle" of a fiber space. Let

$$E = N \times R$$

be the product fiber space, with base R. (Physically, R is thought of as the "time" space). As explained in VB and CPS, M is then identifiable with $J^1(E)$, the space of 1-jets of cross-section maps: $R \to E$.

Definition. A Lagrangian for N, typically denoted by L, is a real-valued (C^∞) function: $M \to R$.

Suppose that $t \to \sigma(t)$, $a \le t \le b$, is a curve in N, and L is a Lagrangian. The action of σ, (relative to L), is the following real number:

$$\mathcal{L}(\sigma) = \int_a^b L(\sigma'(t), t)dt. \tag{2.2}$$

(Recall that $t \to \sigma'(t) \in N_{\sigma(t)}$ is the tangent vector curve to N. It is a curve in T(N). Hence, $t \to (\sigma'(t), t)$ is the "graph" of the tangent vector curve, i.e. a curve in

$$M = T(N) \times R.$$

The "Cartan form" construction assigns to each such Lagrangian function a one-differential form, denoted by $\theta(L)$, on M. This assignment

$$L \to \theta(L)$$

defines a first-order linear differential operator

$$F(M) \to F^1(M).$$

Although this assignment is completely "intrinsic," i.e. independent of local coordinate systems, it is most convenient to use coordinates to define it.

Choose indices and the summation convention as follows:

$$1 \le i, j \le n = \dim N.$$

Suppose that (q_i) is a coordinate system of functions on N. Use these coordinates to define coordinates $(q_i, \dot{q}_i)$ for T(N):

$$\dot{q}_i(v) = dq_i(v)$$

for $v \in T(N)$.

Add the "time" variable "t" to these variables to define

$$(q, \dot{q}, t)$$

as coordinates for $M = T(N) \times R$.

Thus, a Lagrangian function L on M becomes a function $L(q, \dot{q}, t)$ of these variables. The Cartan form $\theta(L)$ is then given by the following formula in these variables:

$$\theta(L) = Ldt + \frac{\partial L}{\partial \dot{q}_i} (dq_i - \dot{q}_i dt). \tag{2.3}$$

In the Caratheodory theory, a special role is assigned to the Lagrangian functions L_S defined by a real valued function

$$S: N \times R \to R. \tag{2.4}$$

In these local coordinates, L_S is defined as follows:

$$L_S(q, \dot{q}, t) = \frac{\partial S}{\partial q_i} (q, t)\dot{q}_i + \frac{\partial S}{\partial t} (q, t). \tag{2.5}$$

For such a Lagrangian, the "action" function (2.2) is readily computable as follows:

$$\mathcal{L}_S(\sigma) = S(\sigma(b), b)) - S(\sigma(a), a). \tag{2.6}$$

<u>Exercise</u>. Show that:

$$\theta(L_S) = \frac{\partial S}{\partial q_i} dq_i + \frac{\partial S}{\partial t} dt = dS. \tag{2.7}$$

3. TIME-DEPENDENT CARATHEODORY VECTOR FIELDS

Keep the notations of Section 2. Consider $M = T(N) \times R$ as a fiber space over $N \times R$, by assigning to a point

$(v,t) \in T(N) \times R$, with $v \in N_p$, $p \in N$, the point

$$(p,t) \in N \times R.$$

A <u>time-dependent vector field on N</u> is then defined as a cross-section map:

$$\alpha: N \times R \to T(N) \times R. \tag{3.1}$$

We will denote such an object by

$$t \to X_t,$$

where, for each t, $X_t \in V(N)$. The cross-section map (3.1) then assigns

$$\alpha(p,\ t) = (X_t(p),\ t) \in T(N) \times R$$

to each pair $(p,\ t) \in N \times R$. (Of course, alternately, one may regard it as a curve in V(N).)

<u>Definition</u>. A curve $t \to \sigma(t)$, $a \le t \le b$, in N is an <u>integral curve</u> of the time-dependent vector field $t \to X_t$ if:

$$\sigma'(t) = X_t(\sigma(t)) \text{ for } a \le t \le b. \tag{3.2}$$

<u>Remark</u>. When made explicit in terms of local coordinates, (3.2) becomes a system of ordinary differential equations of the form:

$$\frac{dq_i}{dt} = \varphi_i(q(t),\ t) \tag{3.3}$$

for the curve $t \to (q_i(t)) = \sigma(t)$, where

$$\varphi_i(p, t) = X_t(q_i)(p) \tag{3.4}$$

for all $p \in N$.

__Definition__. A time dependent vector field

$$\alpha\colon N \times R \to T(N) \times R$$

is a __Caratheodory field__ for the Lagrangian L if the following conditions are satisfied:

$$\alpha^*(L) = 0 \tag{3.5}$$

$$\alpha^*(\theta(L)) = 0. \tag{3.6}$$

Let us see what these conditions mean in local coordinates. Combining (3.4) and (3.5) gives the following condition:

$$L(q, \varphi(q, t), t) \equiv 0 \tag{3.7}$$

(2.3) applies to give:

$$\alpha^*(\theta(L)) = \alpha^*(L)dt + \alpha^*(\frac{\partial L}{\partial \dot{q}_i})(dq_i - \alpha^*(q_i)dt).$$

Since $\alpha^*(L) = 0$, by (3.5), we see that (3.6) is satisfied if and only if:

$$\frac{\partial L}{\partial \dot{q}_i}(q, \varphi(q,t), t) \equiv 0. \tag{3.8}$$

Thus, conditions (3.7) and (3.8) are equivalent to (3.5) and (3.8).

Theorem 3.1. If $t \to \sigma(t)$ is an integral curve of the time-dependent vector field on N defined by the Caratheodory vector field α, then

$$\underset{\sim}{L}(\sigma) = 0, \tag{3.9}$$

i.e. the action along σ vanishes.

Proof. We see from (3.2) that:

$$L(\sigma'(t), t) = \alpha^*(L)(\sigma(t), t),$$

which vanishes by (3.5).

We now ask for conditions which imply that $\underset{\sim}{L}(\sigma) \geq 0$ for all curves σ in N. This involves us with what are called "Legendre conditions" in the classical calculus of variations literature. However, I prefer to call them "convexity conditions."

4. CONVEX LAGRANGIANS

We refer to Rockafellar [1] as the standard mathematical reference on the circle of ideas summarized by the slogan "convexity." Here, we shall only use very elementary ideas, and restrict attention to C^∞ geometric objects. The reader who is familiar with both areas can probably see for himself that putting together the ideas of the classical calculus of variations and the more modern "convexity" ideas is a very productive and rich field.

Let us start with the most elementary notion, a convex real-valued function of a real variable.

<u>Definition</u>. Let $s \to f(s)$, $-\infty < s < \infty$ be a real-valued function of a real variable s. The function is said to be <u>convex</u> if:

$$\frac{d^2f}{ds^2}(s) \ge 0 \text{ for all } s \tag{4.1}$$

It is said to be <u>strictly convex</u> if

$$\frac{d^2f}{ds^2}(s) > 0 \text{ for all } s. \tag{4.2}$$

<u>Theorem 4.1</u>. Suppose that the function $f: R \to R$ is convex. Let $a \in R$ be a <u>critical point</u> of f, i.e.

$$\frac{df}{ds}(a) = 0. \tag{4.3}$$

Then,

$$f(s) \ge f(a) \text{ for all } s \in R. \tag{4.4}$$

If f is strictly convex, then

$$f(s) > f(a) \text{ for all } s \in R - (a). \tag{4.5}$$

<u>Proof</u>. The argument is elementary calculus. (4.1) implies that $s \to \frac{df}{ds}(s)$ is increasing. Hence, (4.3) implies that:

$$\frac{df}{ds}(s) \begin{cases} \ge 0 \text{ for } s \ge a \\ \le 0 \text{ for } s \le a \end{cases} \tag{4.6}$$

(4.6) implies that $s \to f(s)$ is increasing for $s \geq a$, decreasing for $s \leq a$, which implies (4.4). (4.5) follows using similar arguments.

We can now extend the notion of a "convex function" to real valued functions on real vector spaces.

<u>Definition</u>. Let V be a real finite dimensional vector space, and let $f: V \to R$ be a real valued function. f is said to be <u>convex</u> if the following condition is satisfied:

For each $v_1, v_2 \in V$, the function

$$s \to f(v_1 + sv_2), \tag{4.7}$$

which maps $R \to R$, is convex. f is <u>strictly convex</u> if the function (4.7) is strictly convex whenever $v_2 \neq 0$.

<u>Exercise</u>. Let (v_i), $1 \leq i, j \leq n$, be a basis for V. Let (x_i) be the corresponding coordinate system of linear functions on V, i.e.

$$v = x_i(v)v_i$$

for all $v \in V$.

f becomes a function $f(x_1,\ldots,x_n)$ of these variables. Show that f is convex if and only if the $n \times n$ real matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(v)\right) \tag{4.8}$$

is non-negative for all $v \in V$.

(The matrix (4.8) is called the Hessian matrix of f. To say that it is "non-negative" is to say that the quadratic form it defines on R^n takes non-negative values). Show also that f is strictly convex if and only if matrix (4.8) is positive definite.

Remark: Geometrically, this definition means that f is convex if and only if it is convex when restricted to each straight line of V. This definition can obviously be extended to affinely connected manifolds. See DGCV, Chapter 27 and GPS, Chapter 6.

Exercise. Let $f: V \to R$ be a convex function on the finite dimensional vector space V. Let $v \in V$ be a critical point of f, i.e. the following condition is satisfied:

$$\frac{d}{ds} f(v_0 + sv)/_{s=0} = 0 \quad \text{for all } v \in V. \tag{4.9}$$

Show that

$$f(v) \geq f(v_0) \quad \text{for all } v \in V. \tag{4.10}$$

If f is strictly convex, show that

$$f(v) > f(v_0) \quad \text{for all } v \in V - (v_0) \tag{4.11}$$

Now we must deal with modifications - by adding on "linear, inhomogeneous" function to convex functions.

Definition. Let V be a real vector space. A function $h: V \to R$ is said to be a linear inhomogeneous function if there is a constant $c \in R$ and a linear map $\lambda: V \to R$ such that:

$$f(v) = h(v) + c$$

for all $v \in V$.

Exercise. If $f: V \to R$ is convex and if h is linear inhomogeneous, show that f + h is convex. If f is strongly convex show that f + h is strongly convex.

Now return to the calculus of variations. Let N be a manifold, T(N) its tangent bundle, and

$$M = T(N) \times R.$$

Recall that a Lagrangian is a map

$$L: M \to R.$$

We can define a map

$$\pi: M \to N \times R$$

by mapping $(v,t) \in N_p \times R$, with $p \in N$, $t \in R$, into (p,t). The fibers of π are the tangent vector spaces to N, i.e. π defines M as a vector bundle over $N \times R$.

__Definition__. The Lagrangian L is __convex__ if it is convex when restricted to each vector space fiber of π. It is said to be __strictly convex__ if it is strictly convex when restricted to each vector space fiber of π.

__Theorem 4.2__. Let α: $N \times R \to M$ be a Caratheodory field for the Lagrangian L. Suppose also that L is convex. If σ is a curve in N, then

$$\underset{\sim}{L}(\sigma) \geq 0. \qquad (4.12).$$

Further, if L is a strictly convex Lagrangian, then $\underset{\sim}{L}(\sigma) = 0$ if and only if σ is an integral curve of the field φ.

__Proof__. Suppose that, for each $t \in R$, X_t is a vector field on N such that:

$$\alpha(q,t) = (X_t(q),t) \qquad (4.13)$$

for all $g \in N$, $t \in R$.

For q,t fixed, consider the function

$$v \to L(v,t), \qquad (4.14)$$

for $v \in N_q$.

It is a map: $N_q \to R$. By assumption, it is convex relative to the vector space structure on N_q. By (3.8), we see that the function (4.14) has a critical point at

$$v = X_t(q).$$

By (4.10), we see that:

$$L(v,t) \geq 0 \tag{4.15}$$

for all $v \in N_q$.

Hence,

$$\mathcal{L}(\sigma) = \int_a^b L(\sigma'(t),t)dt,$$

which is non-negative, as in an integral of a continuous, non-negative function. This proves (4.12).

Suppose now that the function (4.14) is strictly convex. By (4.11), we see that:

$$\begin{aligned} &L(v,t) > 0 \\ &\text{if } v \neq X_t(q). \end{aligned} \tag{4.16}$$

Suppose $t \to \sigma(t)$, $a \leq t \leq b$, is a curve in N (say, of differentiability class C^1) which is not an integral curve of the field $\{X_t\}$. Then, by (4.16) for some value of t_0, we have:

$$L(\sigma'(t_0),\ t_0) > 0. \tag{4.17}$$

Now, $t \to L(\sigma'(t),t)$ is a continuous function of t (since σ is C^1), which is non-negative for all t, and positive for at least one value t_0. its integral is therefore positive. This implies that:

$$\mathcal{L}(\sigma) > 0.$$

On the other hand, because of condition (3.5) (one of the two conditions defining the notion of "Caratheodory field") we see that

$$\mathcal{L}(\sigma) = 0$$

if:

$$\sigma'(t) = X_t(\sigma(t))$$
for all t,

i.e. if σ is an integral wave of the one-parameter family $\{X_t\}$ of vector fields.

5. EXTREMAL FIELDS AND HAMILTON-JACOBI FUNCTIONS

Let us recapitulate notations. N is a manifold, T(N) is its tangent bundle, M = T(N) × R. Let us now assume that L is a real-valued function on M, but no more.

Suppose that S is a function:

$$S: N \times R \to R.$$

Let L_S be the function: M → R defined by formula (2.5) In other words, L_S is the function on M = T(N) × R such that:

$$\mathcal{L}_S(\sigma) = \int_a^b \frac{d}{dt} S(\sigma(t),t)dt \qquad (5.1)$$

for each curve $t \to \sigma(t)$, $a \le t \le b$, in N.

<u>Definition</u>. A pair (S,α) consisting of a real-valued funcion $S: N \times R \to R$ and a cross-section map

$$\alpha: N \times R \to T(N) \times R$$

is said to define an <u>extremal field</u> for the <u>Lagrangian L</u> if the following condition is satisfied:

$$\begin{array}{l}\alpha \text{ is a Caratheodory field} \\ \text{for the Lagrangian } L - L_S.\end{array} \tag{5.2}$$

Here is the main result:

<u>Theorem 5.1</u>. Suppose that L is a convex Lagrangian, and that (S, α) is an extremal field for L. Let $t \to \sigma(t)$, $a \le t \le b$, be a curve in N. Then,

$$\mathcal{L}(\sigma) \ge S(\sigma(b),b) - S(\sigma(a),a) \tag{5.3}$$

If, further, L is strictly convex, then equality holds in (5.3) if and only if σ is an integral curve for the field (α).

<u>Proof</u>. We see from (5.1) that, for fixed $t \in R$, $q \in N$, the map $v \to L(v,t) - L_S(v,t)$ of $N_q \to R$ is the sum of a convex function and a linear inhomogeneous one, hence it too is convex.

Using Theorem 4.2, we know that

$$\mathcal{L}'(\sigma) \ge 0, \tag{5.4}$$

where $L' = L - L_S$. Using (5.1),

$$\mathcal{L}_S(\sigma) = S(\sigma(b)) - S(\sigma(a)). \tag{5.5}$$

Combining (5.4) and (5.5) proves (5.3).

From Theorem 4.2, we then know that:

$$\mathcal{L}_S(\sigma) \geq 0. \tag{5.4}$$

Let us now calculate $\mathcal{L}_S(\sigma)$ using (5.1):

$$\begin{aligned}\mathcal{L}_S(\sigma) &= \int_a^b (L(\sigma'(t),t) - dS(\sigma'(t),t) \\ &\quad - \frac{\partial S}{\partial t}(\sigma(t),t))dt \\ &= \int_a^b L(\sigma'(t),t)dt - \int_a^b \frac{d}{dt}(S(\sigma(t),t))dt \\ &= \mathcal{L}(\sigma) - S(\sigma(b),b) - S(\sigma(a),a).\end{aligned} \tag{5.5}$$

Combining (5.4) and (5.5) proves (5.3).

To deal with the farther condition that L is strictly convex, again that $L - L_S$ is strongly convex, hence $\mathcal{L}(\sigma) - \mathcal{L}_S(\sigma) > 0$, unless σ is an integral curve of σ.

Definition. A function $S: N \times R \to R$ is a Hamilton-Jacobi Function for the Lagrangian L if there is a cross-section map $\alpha: N \times R \to T(N) \times R$ such that the pair (α,S) is an extremal field.

Let us examine the conditions that α must satisfy in order that there be an S such that (α,S) is an extremal field.

From (3.6), we have:

$$\begin{aligned}0 &= \alpha^*(\theta(L) - \theta(L_S)) \\ &= \alpha^*(\theta(L)) - \alpha^*\pi^*(dS) \\ &\qquad \text{(using 2.7)} \\ &= \alpha^*(\theta(L)) - dS,\end{aligned} \tag{5.6}$$

since $\alpha: N \times R \to T(N) \times R$ is a cross-section map, i.e. $\pi\alpha$ = identity.

To interpret condition (3.5), introduce coordinates (q_i, q_i, t) for M, as in Sections 2 and 3. Then, α is given as follows:

$$\alpha(q,t) = (q, \varphi_i(q,t), t).$$

Hence, (3.5) becomes:

$$L(q,\varphi(q,t), t) = \frac{\partial S}{\partial q_i} \varphi_i(q,t) + \frac{\partial S}{\partial t}. \tag{5.7}$$

In these coordinates, (5.6) means that:

$$\begin{aligned} &\alpha^*(L)dt + \alpha^*(\frac{\partial L}{\partial q_i})(dq_i - \varphi_i\, dt) \\ &= \frac{\partial S}{\partial q_i} dq_i + \frac{\partial S}{\partial t} dt \end{aligned} \tag{5.8}$$

From (5.8), we have:

$$\frac{\partial S}{\partial q_i} = \alpha^*(\frac{\partial L}{\partial \dot{q}_i}). \tag{5.9}$$

Set:

$$H_L = \frac{\partial}{\partial \dot{q}_i} \dot{q}_i - L \tag{5.10}$$

<u>Definition</u>. H_L, defined by formula (5.10), is called the <u>Hamiltonian function</u> of the Lagrangian L.

<u>Remark</u>. In this way of developing the calculus of variations H_L is a real-valued function on the space $T(N) \times R$. In an alternate way of doing it (e.g. see Abraham and Marsden [1]) H_L is a function on $T^d(N) \times R$, where $T^d(N) \times R$ denotes the cotangent bundle to N.

These formulas now prove the following result:

Theorem 5.2. The pair (α,S) is an extremal field for L if and only if the following conditions are satisfied:

$$\frac{\partial S}{\partial t} + \alpha^*(H_L) = 0 \tag{5.11}$$

$$\frac{\partial S}{\partial q_i} = \alpha^*\left(\frac{\partial L}{\partial \dot{q}_i}\right). \tag{5.12}$$

Proof. Equation (5.12) is, of course, identical to (5.9). Note that:

$$\alpha^*(\dot{q}_i) = \varphi_i \tag{5.13}$$

(5.12) now follows from (5.8), (5.9), (5.10) and (5.13).

Conversely, suppose (5.11) and (5.12) are satisfied. One sees readily that they imply (5.8), which implies (5.6), hence that (α,S) is an extremal field for L.

Theorem 5.3. Suppose that (α,S) is an extremal field for the Lagrangian L. Then,

$$\alpha^*(d\theta(L)) = 0. \tag{5.14}$$

Proof. (5.14) follows on applying the exterior derivative operation d to both sides of (5.6).

Remark: Conditions (5.11) form a system of partial differential equations for (α,S) called the Hamilton-Jacobi equation. We shall now show how it can be put into the more traditional form:

In addition to the "configuration space" variables (q_i) for N, introduce a set (p_i) of "momentum variables." Suppose that $H_L'(p,q,t)$ is a real-valued function of the indicated variables such that:

$$H_L'\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q},t),q,t\right) = H_L, \tag{5.15}$$

where H_L is the function of $(q,\dot{q},t)$ defined by (5.10). Notice now that conditions (5.11)-(5.12) imply that:

$$\frac{\partial S}{\partial t} + H_L'\left(q, \frac{\partial S}{\partial q}, t\right) = 0. \tag{5.16}$$

This is the "Hamilton-Jacobi equation" in its traditional form. It is clearly equivalent to conditions (5.11) and (5.12).

6. THE EULER-LAGRANGE EXTREMAL EQUATIONS AND CHARACTERISTIC CURVES

If the Lagrangian is convex, we have seen that the integral curves of extremal fields actually do realize the minumum of the "action." (Theorem 5.1) On the other hand, the curves which minimize the action must satisfy the classical Euler-Lagrange equations. We shall now pursue this direction.

First we review some material about "characteristic curves" of closed two-forms. (See DGCV). Let M be a manifold, and let ω be a closed 2-form on M.

<u>Definition</u>. A curve $t \to \gamma(t)$, $a < t < b$, on M is a <u>characteristic</u>

curve of ω if:

$$\gamma'(t) \rfloor \omega = 0 \tag{6.1}$$

for $a < t < b$.

A vector field $X \in V(M)$ is a characteristic vector field if:

$$X \rfloor \omega = 0 \tag{6.2}$$

A tangent vector $v \in T(M)$ is a characteristic vector of if:

$$v \rfloor \omega = 0. \tag{6.3}$$

Theorem 6.1. $X \in V(M)$ is a characteristic vector field if and only if $X(p)$ is a characteristic vector, for all $p \in M$.

Theorem 6.2. A curve is characteristic if and only if its tangent vector at each point is characteristic.

Theorem 6.3. If $X \in V(M)$ is characteristic, then each integral curve of X is a characteristic curve of ω.

Theorem 6.4. Suppose that the dimension of the space of characteristic vectors of ω is the same at each point of M. Then, each characteristic curve of ω is the integral curve of at least one characteristic vector field.

The proofs of Theorems (6.1)-(6.3) should be obvious. Theorem (6.4) is left as an exercise.

Examples.

a) Hamilton equations

Suppose that $M = R^{2n+1}$. Choose indices $1 \le i, j \le n$, and real variables (q_i, p_i, t) for M. Let $H(p,q,t)$ be a function of these variables. (It is called the Hamiltonian). Set:

$$\omega = d(p_i dq_i - H\, dt). \tag{6.4}$$

Theorem 6.5. There is a unique vector field $X \in V(M)$ which is characteristic for ω and which satisfies the additional condition:

$$X(t) = 1.$$

It is given by the following explicit formula:

$$X = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{6.5}$$

If $v \in T(M)$ is a characteristic vector of ω, then v is a constant multiple of the value of X. Finally, the integral curves of X are precisely the curves in M which are of the form:

$$t \to (q(t),\ p(t),\ t),$$

where $(q(t), p(t))$ are solutions of the following Hamilton equations:

$$\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}(q(t),\ p(t),\ t) \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}(q(t),\ p(t),\ t) \end{aligned} \tag{6.6}$$

The proof of these statements is to be found in DGCV, Chapter 13.

b) Euler-Lagrange Equations

Now, let N be a manifold with variables (q_i), and let:

$$M = T(N) \times R.$$

Choose variables $(q_i, \dot{q}_i, t)$ for M, as explained in Section 2. Let $L(q, \dot{q}, t)$ be a real-valued function on M, i.e. a Lagrangian function for N.

Let $\theta(L)$ be the Cartan form of L, i.e.

$$\theta(L) = L\, dt + \frac{\partial L}{\partial \dot{q}_i}(dq_i - \dot{q}_i\, dt). \tag{6.7}$$

Theorem 6.6. A curve of the form

$$t \to (q(t), \dot{q}(t), t),$$

in M with:

$$\dot{q}_i(t) = \frac{dq_i}{dt}, \tag{6.8}$$

is a characteristic curve of $d\theta(L)$ if and only if:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\left(q(t), \frac{dq}{dt}, t\right)\right) = \frac{\partial L}{\partial q_i}\left(q(t), \frac{dq}{dt}, t\right) \tag{6.9}$$

Again, this is proved in DGCV, Chapter 15. Equation (6.9) are called the Euler-Lagrange equations, for Lagrangian L.

After this review material, let us return to the situation discussed in Section 5. Let N be a manifold.

$$M = T(N) \times R.$$

L a Lagrangian function: $M \to R$. $\theta(L) \in F^1(M)$ its Cartan form.

Consider N as a fiber space over $N \times R$, by mapping (v,t), with $v \in N_q$, into (q,t). Let

$$\alpha: N \times R \to M$$

be a cross-section map. It is then of the form:

$$\alpha(q,t) = (X_t(q),t), \tag{6.10}$$

$$\text{for } q \in N,\ t \in R$$

where (X_t) is a one-parameter family of vector fields on M.

Theorem 6.5. Suppose that:

$$\alpha^*(d\theta(L)) = 0. \tag{6.11}$$

Suppose also that $t \to \sigma(t) \in N$, $a \le t \le b$ is an integral curve of (X_t), in the sense that:

$$\sigma'(t) = X_t(\sigma(t))$$

$$\text{for } a \le t \le b.$$

Then, the curve

$$t \to \gamma(t) = \alpha(\sigma(t),t)$$

in M is a characteristic curve of $d\theta(L)$.

Again, the proof of this result is given in DGCV. It would actually be a good exercise for the reader to prove it for himself.

7. CARATHEODORY EXTREMAL FIELDS FOR VARIATIONAL PROBLEMS WITH CONSTRAINTS

So far, we have been dealing only with the mathematical situation of classical mechanics. However, we shall now show that the methods developed already are powerful enough to deal with the sort of variational problem encountered in optimal control theory.

Let N be a manifold, T(N) its tangent bundle and

$$M = T(N) \times R.$$

As before, consider M as a fiber space over $N \times R$, mapping $(v,t) \in N_q \times R$ into (q,t).

A <u>Lagrangian</u> for N is again a map

$$L: M \to R.$$

If S is a function: $N \times R \to R$, let L_S be the Lagrangian for N with the following property:

If $t \to \sigma(t)$, $a \leq t \leq b$, is a curve in N, then

$$\mathcal{L}_S(\sigma) = S(\sigma(b), b) - S(\sigma(a),a). \tag{7.1}$$

(The definition in local coordinates is given by (2.5).)

In addition to this data, suppose also given a subset

$$C \subset M. \tag{7.2}$$

It will be called in <u>constraint set</u>. The triple (N,L,C) will be said to define a <u>variational problem with constraints</u>. From now on, let us suppose that such a constrained variational problem is given.

Definition. A pair (α,S) consisting of a function $S\colon N \times R \to R$ and a cross-section map $\alpha\colon N \times R \to M$ is called a Caratheodory extremal field for the variational problem (N,L,C) if the following conditions are satisfied:

$$L(v,t) \geq L_S(v,t) \tag{7.3}$$

for all $(v,t) \in C$

$$L(\alpha(q,t)) = L_S(\alpha(q,t)) \tag{7.4}$$

for all $(q,t) \in N \times R$

$$\alpha(N \times R) \subset C. \tag{7.5}$$

Notice that this definition works, no matter what the constraint set. For example, the constraints could be "equalities" (which would correspond to the classical Lagrange variational problem) or "inequalities", which would correspond to the "non-classical" problems considered in modern optimal control and differential games theory. The beauty of these definitions is that they cover all cases simultaneously.

8. CARATHEODORY FIELDS FOR "CLASSICAL" VARIATIONAL PROBLEMS WITH EQUALITY CONSTRAINTS

Continue with the set-up described in Section 7. It is just as convenient to work here with local coordinates for N, labelled

$$(q_i),\ 1 \le i,\ j \le m = \dim N. \qquad (8.1)$$
$$(q_i,\ \dot{q}_i,\ t) \text{ coordinates for } M.$$

Suppose also that $h_a(q,\dot{q},t)$, $1 \le a,\ b \le m$, are functions on M which define the constraints, i.e.

$$C = \{(q,\dot{q},t) \in M\colon h_a(q,\dot{q},t) = 0\} \qquad (8.2)$$

The Lagrangian L is a function $L(q,\dot{q},t)$ of the variables. S is a function $S(q,t)$.

$$L_S = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t}. \qquad (8.3)$$

Definition. The constraints $\{h_a\}$ are regular if the following condition is satisfied:

$$\operatorname{rank}\left(\frac{\partial h_a}{\partial \dot{q}_i}\right) = m. \qquad (8.4)$$

Henceforth, we shall suppose that condition (8.4) is satisfied.

Let us examine the meaning of conditions (7.3) and (7.4). Hold $(q,t) \in N$ fixed. Let

$$C_{(q,t)} = \{v \in T(N)\colon v \in N_q,\ (v,t) \in C\} \qquad (8.5)$$

(In other words, $C_{(q,t)}$ is the intersection of the constraint set C and the fiber in M above the point $(q,t) \in N$). Then, the following condition is satisfied:

The function

$v \to L(v,t) + L_S(v,t)$

has a minimum, as v varies over (8.6)

$C_{(q,t)}$, at the point $v = \alpha(q,t)$

Now,

$$C_{(q,t)} = \{v \in T(N): (v,t) = 0\}. \tag{8.7}$$

Hence, there are, for fixed (q,t), real numbers (λ_a) such that the following condition is satisfied:

The function

$$v \to L(v,t) + L_S(v,t) + \lambda_a h_a(v,t) \tag{8.8}$$

has a critical point at

$$v = \alpha(v,t), \text{ as } v \text{ varies over ALL of } N_q^q.$$

Remark. The numbers λ as defined above are the classical Lagrange multipliers. Let us recall their elementary properties. Let X be a manifold, and let $f: X \to R$ be a real-valued function on X. Let (g_a), $1 \leq 0 \leq n$, be a functionally independent set of real-valued functions on X, and let Y

$$Y = \{x \in X: h_a(x) = 0\}.$$

Then, Y is a submanifold of X. Let $x_0 \in Y$. Then, f restricted to Y, if Y has a critical point at X_0(i.e. the exterior derivative $d(fy)$ is zero at x_0, if and only if there are numbers $\lambda_a \in R$ such that:

$$df(x_0) = -\lambda_a dh_a(x_0),$$

i.e. x_0 is a critical point of $(f + \lambda_a g_a)$.

Return to condition (8.8). As (q,t) runs over $N \times R$, the λ_a become functions $\lambda_a(q,t)$ of these variables.

Exercise. Show that, as a consequence of condition (8.4) (and the implicit function theorem) the Lagrange multipliers $\lambda_a(q,t)$ are C^∞ functions of (q,t).

Let us return to the use of local coordinates $(q,\dot{q},t)$. We can then express condition (8.8) as follows:

$$0 = \frac{\partial L}{\partial \dot{q}_i}(\alpha(q,t)) + \frac{\partial S}{\partial q_i}(q,t) + \lambda_a(q,t)\frac{\partial h}{\partial \dot{q}_i}. \qquad (8.9)$$

Condition (7.4) is:

$$L(\alpha\ (q,t)) - L_S(\alpha(q,t)) = 0 \qquad (8.10)$$

Condition (7.5) is:

$$h_a(\alpha(q,t)) = 0 \qquad (8.11)$$

Hence,

$$\alpha^*(\theta(L) - \theta(L_S) + \lambda_a\ \theta(g_a)) = 0. \qquad (8.12)$$

But,

$$\alpha^*(\theta(L_S)) = dS. \qquad (8.13)$$

This proves the following main result:

Theorem 8.1. If the pair (α,S) defines a Caratheodory extremal field for the constrained variational problem, then there are functions $\lambda_a(q,t)$ such that:

$$\alpha^*(\theta(L)) - dS = -\varphi_a\alpha^*(\theta(h_a)). \qquad (8.14)$$

In particular, after applying d to both sides of (8.14), we have the following relations:

$$\alpha^*(d(\theta(L + \lambda_a h_a)) = 0 \tag{8.15}$$

or

$$\alpha^*(d\theta(L) + d(\lambda_a \theta(h_a)) = 0. \tag{8.16}$$

Equations (8.15) or (8.16) (they are in fact readily seen to be equivalent) are the partial differential equation determining the Caratheodory extremal fields in this "constrained" case.

Our next goal is to see the relation between conditions (8.15) or (8.16) and Hamilton-Jacobi theory. To this end, consider the space of variables $(q_i, \dot{q}_i, t, \lambda_a)$. Consider the set of points in this space at which the following relations are satisfied:

$$h_a(q,\dot{q},t) = 0. \tag{8.17}$$

Call this space M'. Thus,

$$\dim M' = 2n+1.$$

Let ω be the closed 2-form on M' defined by the following formula:

$$\omega = d(\theta(L) + \lambda_a \theta(h_a)). \tag{8.18}$$

Thus, ω is the form defined by the formula on the right hand side of (8.18), then restricted to M', i.e. bound by the relation (8.17).

Suppose now that (α,S) is a Caratheodory extremal field for the Lagrange variational problem defined by the Lagrangian L and the constraint functions h_a. Let $\lambda_a(q,t)$ be the functions: $N \times R \to R$ which satisfy (8.14).

Suppose α, the cross-section maps: $N \times R \to T(N) \times R$, is given in coordinates by the following formulas:

$$\alpha(q,t) = (q,\varphi_i(q,t),t).$$

We can then define a mapping

$$\alpha'\colon N \times R \to M'$$

by the following formula

$$\alpha'(q,t) = (q, \varphi_i(q,t), t,\lambda_a(q,t)). \tag{8.19}$$

Condition (8.16) now means that:

$$\alpha'^*(\omega) = 0, \tag{8.20}$$

i.e. α' is an integral manifold of the 2-form ω on M'. Further, the following "trans versality" conditions are satisfied.

The map α' is transversal to the function (q_i,t) on M', in the sence that the pull-back form (8.21)

$$\alpha'^*(d_{q_i}),\alpha'^*(dt)$$

are functionally independent.

Condition (8.20) and (8.21) obviously provide us with an alternate way of considering the condition for Caratheodory extremal field which is directly linked to Hamilton-Jacobi theory as explained in DGCV.

An alternate way of relating the study of extremal fields to classical Hamilton-Jacobi theory (as presented, for example, in Caratheodory's book [1]) is via the notion of the "canonical form" of the closed 2-form ω. Namely, a set of functions $(x_1,\ldots,x_m, y_1,\ldots, y_m, H(x,y,t))$ on M' such that:

$$\omega = dy_1 \wedge dx_1 + \ldots + dy_m \wedge dx_m + dH \wedge dt. \qquad (8.2)$$

In these coordinates, everything takes its form described in Caratheodory's book. We shall now turn to the description of these coordinates, in the cases of prime interest in optimal control-systems theory.

9. VARIATIONAL PROBLEMS IN CONTROL AND SYSTEMS THEORY

Let us completely change notations, in order to conform more closely with the notations in treatises on control and systems theory. (For example, see Brockett [1] and Desoer [1]). Choose indices as follows:

$$1 \le i, j \le n$$
$$1 \le a, b \le m.$$

$i,j,\ldots$ are the <u>state indices</u>, while $a,b,\ldots$ are the <u>control indices</u>.

Introduce variables (x_i), and (u_a). In the way of thinking that is now standard in the system theory literature, the vector $x = (x_i) \in R^n$ is called a <u>state vector</u>, while $u = (u_a) \in R^m$ is called a <u>control vector</u>.

<u>Remark</u>: For our purposes I believe that a more general formulation of the "global" situation is desirable. We shall develop such ideas as we go along.

Let us now consider constraints of the following form:

$$\frac{dx_i}{dt} = f_i(x(t), u(t), t). \tag{9.1}$$

In (9.1), the $f_i(\ ,\ ,\ )$ are real-valued functions of the indicated variables (i.e. maps $R^n \times R^m \times R \to R$.) We can also write (9.1) in a "vectorial" notation:

$$\frac{dx}{dt} = f(x(t), u(t), t). \tag{9.2}$$

Let us think of these equations in the following way. A curve $t \to u(t)$ is an <u>input</u> to the system. A vector $x^\circ \in R^n$ and an input curve $t \to u(t)$, given for $t \geq t_0$, a fixed time, determines (via the usual existence - uniqueness theorem for ordinary differential equations) a curve $t \to (x(t), u(t))$ which is a solution of (9.2), such that:

$$x(t_0) = x^0.$$

In the standard formulation of systems theory, (see Zadeh and Desoer [1] and Desoer [1]) this curve is not given any particular name; I believe that its role should be formalized better. Instead, the usual theory postulates a map r: $(x,u,t) \to y$, which converts the curve $t \to (x(t), u(t),t)$ into a curve $t \to y(t)$ in a new space, called the <u>output curve</u>. The function r is called (for example, by Dosoer [1]) the <u>read-out function</u>.

At the moment, however, I want to put aside these interesting questions of the general formulation of "systems theory," and think of it merely as an interesting mathematical situation. Let $L(x,u)$ be a function of these variables. Suppose that the problem is to extremize

$$\int L(x(t), u(t)dt \tag{9.3}$$

over the space of the curves

$$t \to (x(t), u(t),t)$$

which satisfy (9.2).

The procedure for finding the extremal fields has been described in previous sections. Let us work it out, in this situation:

Let θ_L be the Cartan form for L. Since L does not depend on $\dot{x}$, $\dot{u}$, we have, very simply:

$$\theta_L = Ldt. \tag{9.4}$$

Set:

$$\theta_i = dx_i - f_i\, dt. \tag{9.5}$$

Introduce "Lagrange multiplier" variables λ_i, and set:

$$\begin{aligned}\theta &= L dt + \lambda_i \theta_i \\ &= L dt + \lambda_i (dx_i - f_i\, dt) \\ &= (L - \lambda_i f_i) dt + \lambda_i dx_i . \end{aligned} \tag{9.6}$$

$$\omega = d\theta . \tag{9.7}$$

Let M be the space of variables (x,u,λ,t). It is then $2n+m+1$ -dimensional. θ and ω are differential forms on M.

We shall first see that the notion of "extremal field" developed in previous sections is not really adequate to cover this important situation. To see this, let N be the space of variables (x,u,t), and consider M as a fiber space over N, by mapping

$$(x,u,\lambda,t) \to (x,u,t).$$

Let α be a cross-section map: $N \to M$, and S a real-valued function: $N \to R$, such that:

$$\alpha^*(\theta) = dS \tag{9.8}$$

In view of (9.6), this means that there are functions (x $(x,u,t) \to \lambda_i(x,u,t)$ such that:

$$\frac{\partial S}{\partial t} = L - \lambda_i(x,u,t) f_i \tag{9.9}$$

$$\frac{\partial S}{\partial x_i} = \lambda_i \tag{9.10}$$

$$\frac{\partial S}{\partial u_a} = 0. \tag{9.11}$$

These conditions imply that S and λ are functions of (x,t) alone, which contradicts (9.9). Therefore, this is not an adequate notion of "extremal field." The trouble is that the problem is "singular," in the sense that the 2-form ω has too low a rank to permit the existence of integral manifolds of dimension $n + m + 1$. Hence, we must modify the definition of "extremal field."

The way to do this is to redefine N to be the space of variables (x_i,t). Consider M to be a fiber space over N, mapping

$$(x,u,\lambda,t) \text{ into } (x,t).$$

Define an extremal field now as a pair (α,S) consisting of a cross-section map

$$\alpha\colon N \to M$$

and a function $S\colon N \to R$ such that:

$$\alpha^*(\theta) = dS. \tag{9.12}$$

This means that α is defined by functions $\lambda_i(x,t), u_a(x,t)$ (9.6) and (9.12) together imply that:

$$\frac{\partial S}{\partial t}(x,t) = L(x,u(x,t),t) - \lambda_i(x,t)f_i(x,u(x,t),t) \tag{9.13}$$

$$\frac{\partial S}{\partial x_i} = \lambda_i. \tag{9.14}$$

Hence, (9.13) and (9.14) together imply the following conditions:

$$\frac{\partial S}{\partial t}(x,t) = L(x,u(x,t),t) - \lambda_i(x,t)f_i(x,u(x,t),t) \tag{9.15}$$

We can now show how (9.15) implies that $S(x,t)$ satisfies a Hamilton-Jacobi equation of conventional type. Set:

$$H(x,\lambda,t) = L(x,u(x,t),t) - \lambda_i(x,t)f_i(x,u(x,t),t) \tag{9.16}$$

Then, (9.15) means that:

$$\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x_1}\ t) = 0, \tag{9.17}$$

which is indeed the "Hamilton-Jacobi equation" for S, with H the "Hamiltonian."

Notice that the condition (9.12) does not suffice to determine the functions $u_a(x,t)$. This is a consequence of the "singular" nature of the problem. The "Pontrjagin principle" fills in this gap. With u unspecified, set:

$$H(x,u,\lambda,t) = L(x,u,t) - \lambda_i f_i(x,u,t). \tag{9.18}$$

Then, for fixed (x,t), choose "control law functions" $u_a(x,t)$ so that:

$$H(x,\lambda,t) = \min_u H(x,u,\lambda,t). \tag{9.19}$$

Notice further that (9.19) allows the possibility of choice of constraints on the control variables u_a. The "minimum" on the right hand side of (9.19) now means the "minimum over all allowed values of u."

Later on, I shall discuss very general variational problems from the point of view of the theory of exterior differential systems. In this way, we aim towards a synthesis of techniques of the classical calculus of variations (and classical mechanics) and optimal control theory. We shall also be able to treat field-theoretic problems with the same machinery.

Chapter V

OPTIMAL CONTROL VARIATIONAL PROBLEMS

1. INTRODUCTION

Much of the engineering discipline called "optimal control theory" is, from one point of view, just a special case of the classical calculus of variations. However, there are special features which are very interesting, and which are closely related to important "systems-theoretic" ideas. In this chapter, we shall develop the interrelation between optimal control theory as it is developed in the engineering literature (e.g. see Anderson and Moore [1], Brockett [1], Pontrjagin, Boltjanski and Gamkrelidze [1]) and the "geometric" theory of the Lagrange variational problem (e.g. see DGCV, Caratheodory [1], and my paper [3].

Let us begin by formulating the optimal control variational problem, in a way that is standard in the systems-theory literature, e.g. the work of Brockett, Desoer, Kalman and Zadeh. This description emphasizes the role of the "input", "output" and "state" spaces. From our point of view, it is a "local" description of the situation, and will eventually be replaced by a more "global" fiber-space framework.

Two spaces, X and U, are given, called the <u>state</u> and <u>input</u> (or <u>control</u>) space, respectively. Although for the general formulation of the ideas (see Zadeh and Desoer [1]) the possibility of X and U being quite general sort of spaces should be left open, we shall suppose that X and U are manifolds, with coordinate systems:

$$(x_i),\ 1 \le i,\ j \le n = \dim X$$

$$(u_a),\ 1 \le a,\ b \le m = \dim U.$$

Also, introduce R, the real numbers, parameterized by t, $-\infty < t < \infty$. Physically, the points of R denote "time". Set:

$$M = X \times U \times R$$

$$N = X \times R.$$

Map $M \to N$ using the Cartesian projections, i.e. mapping

$$(x, u, t) \to (x, t).$$

This makes M into a fiber space over N, with the "control space" U as the fiber.

<u>Remark</u>: As we shall see, this is probably a good way to "axiomatize" or "geometrize" systems theory, i.e. to start off with such a fiber space map, not necessarily assuming it to be a product.

<u>Definition</u>. A <u>Lagrangian (of the optimal control type)</u> for the system defined by state space X and input space U is a real valued function

$$L: M \to R.$$

<u>Definition</u>. An <u>input-state</u> relation for the system is defined by a Pfaffian system P on M which, in the coordinates (x_i, u_a, t), has F(M)-bases of the following form:

$$\omega_i = dx_i - f_i(x, u, t)dt, \tag{1.1}$$

where the $f_i(x, u, t)$ are real-valued functions on M.

For short, we call this an <u>input-state Pfaffian system</u>.

<u>Remark</u>: The integral curves of P provide a mechanism for defining a mapping, which assigns to a curve $t \to u(t)$ U and a point $x^0 \in X$ a curve in $t \to (x(t), u(t), t)$ in M which is an integral curve of P such that

$$x(0) = x^0.$$

This sort of a mapping is characteristic of an "axiomatization" of systems theory.

These are the basic differential geometric objects which play a role in the material to be developed in this section.

2. VARIATIONAL SYSTEMS OF OPTIMAL CONTROL TYPE

Let N, M, X, U, R, x_i, u_a, t be as defined in Section 1. A variational system is defined by a pair

$$(L, P)$$

consisting of a Lagrangian function

$$L: M \to R$$

and an input-state Pfaffian system P, with a bases (ω_i) given by formula (1.1).

Suppose such a variational system is given. Given real numbers t_0, t_1 with $t_0 < t$, let

$$\pi(P, t_0, t_1)$$

denote the space of curves of the following form:

$$t \to (x(t), u(t), t), \qquad (2.1)$$
$$t_0 \le t \le t_1,$$

which are integral curves of P, i.e. satisfy the following relations:

$$\frac{dx_i}{dt} = f_i(x(t), u(t), t). \qquad (2.2)$$

The Lagrangian L defines a real valued function

$$L: \pi(P, t_0, t_1) \to R$$

called the action function. Here is the formula:

$$L(\sigma) = \int_{t_0}^{t_1} L(\sigma(t))dt \tag{2.3}$$

for $\sigma \in \pi(P, t_0, t_1)$.

With curve σ defined in local coordinates by $t \to (x(t), u(t), t)$, (2.3) takes the following form:

$$\int_{t_0}^{t_1} L(x(t), u(t), t)dt. \tag{2.4}$$

Our job is now to describe the curves $t \to (x(t), u(t))$ in $X \times U$ which "extremize" this integral. Notice that this is a special case of what is called the "Lagrange variational problem" in the classical calculus of variations literature. Accordingly, we shall pause to develop the differential equations of the extremals from the classical point of view, emphasizing the role of the "Lagrange multipliers."

3. EXTREMALS ASSOCIATED WITH LAGRANGE MULTIPLIERS

Keep the notations of Section 2. In addition to the variables (x, u, t), introduce Lagrange multiplier variables (λ_i). Also, introduce velocity variables $\dot{x}_i$.

Set:

$$\begin{aligned} L'(x, \dot{x}, u, \lambda, t) &= L(x, u, t) \\ &+ \lambda_i(\dot{x}_i - f_i(x, u, t)) \end{aligned} \tag{3.1}$$

Definition. Regard (3.1) as defining a Lagrangian for the "configuration space" with variables (x, u, λ). Then, a curve $t \to (x(t), u(t), t)$ in M is an extremal curve of the optimal control variational problem defined by L if there is a curve $t \to (x(t), u(t), \lambda(t))$ which is an extremal (in the unconstrained sense) of the Lagrangian L'.

Let us now work out the conditions that a curve $t \to (x(t), u(t), \lambda(t))$ be an extremal of L'. Of course, this amounts to saying that the curve is a solution of the Euler-Lagrange equations. But, the "velocity" variables $\dot{x}_i$, $\dot{u}$ do not occur in (3.1). Hence Euler-Lagrange equations specialize, in this case, as follows: The first set are:

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\lambda}_i}\right) = \frac{\partial L'}{\partial \lambda_i}, \tag{3.2}$$

but $\frac{\partial L'}{\partial \dot{\lambda}} = 0$, $\frac{\partial L'}{\partial \lambda_i} = \dot{x}_i - f_i(x, u, t)$, hence (3.2) is equivalent to the following conditions:

$$\frac{dx_i}{dt} - f_i(x(t), u(t), t) = 0, \tag{3.3}$$

which are, of course, just the constraint equations (2.2). The second set are:

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}_i}\right) = \frac{\partial L'}{\partial x_i}$$

or, using (3.1),

$$\frac{d}{dt}\lambda_i(t) = \frac{\partial L}{\partial x_i}(x(t), u(t), t) - \lambda_j(t)\frac{\partial f_j}{\partial x_i}(x(t), u(t), t). \tag{3.4}$$

The third set are:

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial u_a}\right) = \frac{\partial L'}{\partial u_a},$$

or

$$0 = \frac{\partial L}{\partial u_a}(x(t), u(t), t) - \lambda_i(t)\frac{\partial f_i}{\partial u_a}(x(t), u(t), t). \tag{3.5}$$

Thus, conditions (3.3)-(3.5) are the equations that a Lagrange multiplier extremal must satisfy.

Let us now show how these equations may put into Hamiltonian form. Set:

$$H'(x, u, \lambda, t) = L(x, u, t) - \lambda_i f_i(x, u, t). \tag{3.6}$$

Let M' denote the space of variables

$$(x, \dot{x}, u, \lambda, t).$$

H' defined by formula (3.6) then is a real-valued function on M'. Consider the functions on M' defined by the following formula:

$$g_a = \frac{\partial H'}{\partial u_a} \tag{3.7}$$

Set:

$$M'' = \{(x, \dot{x}, u, \lambda, t) \in M' : g_a(x, u, \lambda, t) = 0$$
$$\text{and } \dot{x}_i = f_i(x, u, t)\}. \quad (3.8)$$

Equation (3.5) means that the extremal curves all be on the subset M'' of M' defined by formula (3.8).

Let us now investigate the meaning of Equations (3.3) and (3.4). To this end, set:

$$\theta = \frac{\partial L'}{\partial x_i}(dx_i - \dot{x}_i dt) + \frac{\partial L'}{\partial u_a}(du_a - \dot{u}_a dt)$$
$$+ \frac{\partial L'}{\partial \lambda_i}(d\lambda_i - \dot{\lambda}_i dt) \quad (3.9)$$
$$+ L'dt.$$

This formula tells us that θ is the <u>Cartan form</u> of the Lagrangian L'. (Notice that we regard (x, u, λ) as the "configuration space" variables for the Lagrangian L'). Using formula (3.1) for L', we have the following more explicit formula for θ:

$$\frac{\partial L'}{\partial x_i} = \lambda_i$$

$$\frac{\partial L'}{\partial u_a} = 0 = \frac{\partial L'}{\partial \lambda_i}.$$

Hence,

$$\theta = \lambda_i(dx_i - \dot{x}_i dt) + (L + \lambda_i(\dot{x}_i - f_i)dt \quad (3.10)$$

$$= \lambda_i dx_i + H'dt. \quad (3.11)$$

Here is a main result:

<u>Theorem 3.1</u>. A curve $t \to \sigma(t)$ in M" of the form $t \to (x(t), u(t), \lambda(t), t)$ is a Lagrange multiplier extremal curve if and only if it is a characteristic curve of the 2-form $d\theta$.

<u>Proof</u>. Recall (from DGCV) that σ characteristic means that:

$$\sigma'(t) \lrcorner\, d\theta = 0. \tag{3.12}$$

Now,

$$\begin{aligned}\sigma'(t) \lrcorner\, d\theta &= \sigma'(t) \lrcorner\, (d\lambda_i \wedge dx_i + dH' \wedge dt) \\ &= (\tfrac{d}{dt} \lambda_i(t))dx_i - (\tfrac{d}{dt} x_i(t))d\lambda_i \\ &+ \tfrac{d}{dt} (H'(x(t), u(t), \lambda(t), t))dt \\ &- dH'.\end{aligned} \tag{3.13}$$

But,

$$dH' = \frac{\partial H'}{\partial x_i} dx_i + \frac{\partial H'}{\partial u_a} du_a + \frac{\partial H'}{\partial \lambda_i} d\lambda_i + \frac{\partial H'}{\partial t} dt \tag{3.14}$$

Combine (3.12) and (3.13):

$$\begin{aligned}\sigma'(t)d\theta &= (\tfrac{d}{dt} \lambda_i - \tfrac{\partial H'}{\partial x_i})dx_i \\ &+ (\tfrac{\partial H'}{\partial \lambda_i} - \tfrac{d}{dt} x_i)d\lambda_i \\ &+ (\tfrac{\partial H'}{\partial u_a} du_a \\ &+ (\tfrac{\partial H'}{\partial x_i} \tfrac{dx_i}{dt} + \tfrac{\partial H'}{\partial u_a} \tfrac{du_a}{dt} + \tfrac{\partial H'}{\partial \lambda_i} \tfrac{d\lambda_i}{dt})dt.\end{aligned} \tag{3.15}$$

Now, σ lies on the subset M" defined by formula (3.8). In particular,

$$\frac{\partial H'}{\partial u_a}(\sigma(t)) = 0. \tag{3.16}$$

Combine (3.14) and (3.15):

$$\begin{aligned} \sigma'(t)d\theta = {} & \left(\frac{d}{dt}\lambda_i - \frac{\partial H'}{\partial x_i}\right)dx_i \\ & + \left(\frac{\partial H'}{\partial \lambda_i} - \frac{d}{dt}x_i\right)d\lambda_i \\ & + \left(\frac{\partial H'}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial H'}{\partial \lambda_i}\frac{d\lambda_i}{dt}\right)dt. \end{aligned} \tag{3.17}$$

Notice that the right hand side of (3.16) vanishes if and only if the following Hamilton-type equations are satisfied:

$$\begin{aligned} \frac{d}{dt}\lambda_i &= \frac{\partial H'}{\partial x_i}(x(t), u(t), \lambda(t), t) \\ \frac{d}{dt}x_i(t) &= -\frac{\partial H'}{\partial \lambda_i}(x(t), u(t), \lambda(t), t). \end{aligned} \tag{3.18}$$

Use (3.6):

$$\frac{\partial H'}{\partial x_i} = \frac{\partial L}{\partial x_i} - \lambda_j \frac{\partial f_j}{\partial x_i} \tag{3.19}$$

$$\frac{\partial H'}{\partial \lambda_i} = -f_i. \tag{3.20}$$

Hence, equations (3.17) are equivalent to the following equations:

$$\frac{d}{dt}x_i(t) = f_i(x(t), u(t), t) \tag{3.21}$$

$$\frac{d}{dt}\lambda_i(t) = \frac{\partial L}{\partial x_i}(x(t), u(t), t)$$
$$- \lambda_j \frac{\partial f_j}{\partial x_i}(x(t), u(t), t). \qquad (3.22)$$

But, equations (3.20) and (3.21) are precisely the equations (3.3) and (3.4), i.e. (3.20) and (3.21), together with the condition that σ lies in M", are equivalent to the Euler-Lagrange extremal equations for the Lagrangian L'. This completes the proof of Theorem 3.1.

As a bonus, we have also proved the following result, concerning the Hamiltonian form of the Lagrange multiplier extremal equations.

<u>Theorem 3.2</u>. A curve $t \to (x(t), u(t))$ in $X \times U$ is a Lagrange multiplier extremal if and only if there is a curve $t \to \lambda(t)$ in the Lagrange multiplier space such that:

$$\begin{aligned} \frac{dx_i}{dt} &= -\frac{\partial H'}{\partial \lambda_i}(x(t), u(t), \lambda(t), t) \\ \frac{d\lambda_i}{dt} &= \frac{\partial H'}{\partial x_i}(x(t), u(t), \lambda(t), t), \end{aligned} \qquad (3.23)$$

and the following condition is satisfied:

$$\frac{\partial H'}{\partial u_a}(x(t), u(t), \lambda(t), t) = 0. \qquad (3.24)$$

This way of expressing the extremal equations directly ties in our work to that centered around the concept of the "Pontrjagin minimal condition." We now investigate this point.

4. THE PONTRJAGIN PRINCIPLE

Continue with the notations used in Section 3. Recall that H' is a function of the variables (x, u, λ, t). Define a function H of the variables (x, λ, t) by the following formula:

$$H(x, \lambda, t) = \min_{u \in U} H'(x, u, \lambda, t). \tag{4.1}$$

In words, $H(x, \lambda, t)$ is the minimal value taken by H' on fiber above (x, λ, t), for the fiber space mapping:

$$(x, u, \lambda, t) \to (x, \lambda, t)$$

of

$$X \times U \times \Lambda \times R \to X \times \Lambda \times R$$

<u>Definition</u>. A curve $t \to (x(t), \lambda(t))$ in $X \times \Lambda$ is a <u>Pontrjagin extremal of the variational problem</u> if the following equations are satisfied:

$$\begin{aligned} \frac{dx_i}{dt} &= -\frac{\partial H}{\partial \lambda_i}(x(t), \lambda(t), t) \\ \frac{d\lambda_i}{dt} &= \frac{\partial H}{\partial x_i}(x(t), \lambda(t), t). \end{aligned} \tag{4.2}$$

Theorem 4.3. Suppose that there are functions $(x, \lambda, t) \to u(x, \lambda, t)$ such that:

$$H(x, \lambda, t) = H'(x, u(x, \lambda, t), \lambda, t). \tag{4.3}$$

Then, every Pontrjagin extremal is also a Lagrange multiplier extremal.

Proof. Combining (4.2) and (4.3), we see that:

$$\frac{\partial H'}{\partial u_a}(x, u(x, \lambda, t), \lambda, t) = 0. \tag{4.4}$$

Suppose that $t \to (x(t), \lambda(t))$ is a curve satisfying (4.2). Set:

$$u(t) = u(x(t), \lambda(t), t). \tag{4.5}$$

Then, using (4.4), we see that the curve $t \to (u(t), x(t), \lambda(t), t)$ satisfies (3.23).

Let us now show that this curve satisfies (3.22). Then, assuming (4.2),

$$\begin{aligned}\frac{dx_i}{dt} &= -\frac{\partial H}{\partial \lambda_i}(x(t), \lambda(t), t)\\ &=\text{, using (4.3),}\\ &\quad -\frac{\partial H'}{\partial u_a}(x(t), u(t), \lambda(t), t)\frac{du_a}{dt}\\ &\quad -\frac{\partial H'}{\partial \lambda_i}(x(t), u(t), \lambda(t), t),\\ &= -\frac{\partial H'}{\partial \lambda_i}(x(t), u(t), \lambda(t), t) \text{ using (4.4).}\end{aligned}$$

This is the first half of equations (3.22). The second half is proved similarly.

Remark: Our argument assumes that u varies over a manifold U, so that the condition that a function of U take its minimum at a point implies that the partial derivatives vanish at that point. (This was used to derive (4.4)). The Pontrjagin principle also applies to more general problems, e.g. those where u varies over a subset of R^m subject to inequality constraints.

5. THE ORDINARY VARIATIONAL PROBLEM FROM THE OPTIMAL CONTROL POINT OF VIEW

By the "ordinary problem" we mean the unconstrained situation dealt with in all the books on the calculus of variations. Its theory is more-or-less identical with classical mechanics.

Let X be a manifold of dimension n, with a coordinate system (x_i), $1 \le i, j \le n$. Let T(X) denote its tangent vector bundle, and

$$M = T(X) \times R.$$

Consider the time parameter, t, as the variable on R. Let $\dot{x}_i$ be the linear functions on T(X) such that:

$$\dot{x}_i(v) = dx_i(v) \tag{5.1}$$

for $v \in T(X)$.

Then, one readily sees that $(x_i, \dot{x}_i, t)$ form a coordinate system for M. A <u>Lagrangian</u> is a mapping

$$L: M \to R.$$

Locally, it is then given by a function

$$L(x, \dot{x}, t)$$

of these variables. The classical variational problem is then to extremize

$$\int L(x(t), \frac{dx}{dt}, t)dt$$

over all curves $t \to x(t)$ in X. (In classical mechanics, X is called the <u>configuration space</u>).

Let us now convert it into a control theory problem. Regard X as the "state space," and regard $\dot{x}_i$ as the "control variables," i.e. define the control variables (u_i) so that:

$$u_i = \dot{x}_i.$$

Choose the differential equations constraining the curves $t \to (x(t), u(t), t)$ in M as follows:

$$\frac{dx_i}{dt} = u_i. \tag{5.2}$$

Following the formulas developed in previous sections, introduce the new Lagrange multiplier variables (λ_i), and the function:

$$H' = L(x, u, t) - \lambda_i u_i. \tag{5.3}$$

Let M' be the space of variables (x, u, t, λ). Consider the subspace M" of M' defined by the following conditions:

$$\frac{\partial H'}{\partial u_i} = 0. \tag{5.4}$$

Let us calculate M" more explicitly. To this end, define functions $y_i(x, u, t)$ by the following formula:

$$y_i = \frac{L}{u_i}(x, u, t). \tag{5.5}$$

(In the classical mechanics situation, the y_i are the momentum functions). Combine (5.3) and (5.5):

$$\begin{aligned}\frac{\partial H'}{\partial u_i} &= \frac{\partial L}{\partial u_i} - \lambda_i \\ &= p_i - \lambda_i.\end{aligned}$$

Thus, <u>on the subspace M"</u>,

$$\lambda_i = p_i.$$

<u>Definition</u>. The (ordinary) variational problem defined by the Lagrangian L is said to be <u>regular</u> if the following condition is satisfied:

$$\det\left(\frac{\partial^2 L}{\partial u_i \partial u_j}\right) \neq 0. \tag{5.6}$$

Exercise. Show that condition (5.6) is equivalent to saying that the functions

$$(x_i,\ p_i,\ t) \tag{5.7}$$

form a new coordinate system for M.

Let us suppose that (5.6) is satisfied. In these new "momentum" coordinates (5.7), M" is determined by the conditions:

$$\lambda_i = p_i, \tag{5.8}$$

i.e. M" is the linear subspace of $R^{3n+1} = M'$ determined by (5.8) $(x_i,\ p_i,\ t)$ form a coordinate system for M". Then set:

$$H = H' \text{ restricted to } M''.$$

Explicitly, from (5.3), we have:

$$H = L - p_i u_i = L(x,\ \dot{x},\ t) - \lambda_i \dot{x}_i . \tag{5.9}$$

One recognizes that this is (up to a sign) the classical formula for the Hamiltonian function.

Remark: We have chosen our notations so that the Hamiltonian is given by (5.9), instead of its negative, as in the classical theory for esthetic reasons, since I feel it is better that the Pontrjagin principle be a "minimum principle" rather than a "maximal principle."

<u>Exercise</u>. Complete the proof that the general optimal control theory reduces to the classical mechanics theory by showing that the extremals, defined as solutions of (3.22) and (3.23), are identical to the solutions of the Hamilton equations in this classical form:

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = \frac{\partial H}{\partial x_i}.$$

6. HIGHER DERIVATIVE LAGRANGIANS

As application of the formalism developed in the previous sections, let us develop the calculus of variations for higher-derivative Lagrangians, using the "control theory" formalism.

Classically, one is given a "configuration space" with variables

$$q = (q_a),\ 1 \le a \le m,$$

and a Lagrangian function

$$L(q, \frac{dq}{dt}, \frac{d^2q}{dt^2}, \ldots).$$

Introduce "Newtonian" variables

$$(q_a,\ \dot{q}_a,\ \ddot{q}_a,\ldots)$$

Then, L becomes a function of these variables.

For simplicity, we shall suppose in this section that L depends only on the variables $(q,\ \dot{q},\ \ddot{q})$. Problems involving derivatives higher than the second order and time dependence may be handled with similiar methods. Introduce the following range of indices and variables:

$$1 \le i,\ j \le n = 2m$$

$$x_1 = q_1,\ldots,\ x_m = q_m$$

$$x_{m+1} = \dot{q}_1,\ldots,\ x_{2m} = \dot{q}_m$$

$$u_1 = \ddot{q}_1,\ldots,\ u_m = \ddot{q}_m.$$

The constraint differential equations then take the following form:

$$\begin{aligned} \frac{dx_i}{dt} &= x_{m+i} \text{ for } 1 \le i \le m \\ \frac{dx_{m+a}}{dt} &= u_a \text{ for } 1 \le a \le m. \end{aligned} \tag{6.1}$$

$L(q,\ \dot{q},\ \ddot{q})$ then becomes a function of the variables $(x,\ u)$, which we denote by $L(x,\ u)$.

Equations (6.1) may be written in the form:

$$\frac{dx_i}{dt} = f_i(x,\ u), \tag{6.2}$$

with

$$f_a = x_{m+a} \tag{6.3}$$

$$f_{m+a} = u_a. \tag{6.4}$$

Introduce additional Lagrange multiplier variables λ_i. Apply formula (3.6).

$$H' \equiv L - \lambda_i f_i$$

$$= L - \lambda_a f_a - \lambda_{m+a} f_{m+a}$$

$=$, using (6.3)-(6.4),

$$L - \lambda_a x_{m+a} - \lambda_{m+a} u_a. \tag{6.5}$$

Let M be the space of variables

$$(x, u, \lambda, t).$$

H is then a function on M. (In fact, it is independent of t, of course.)

Copy formula (3.7):

$$g_a \equiv \frac{\partial H}{\partial u_a},$$

Using formula (6.5) we have:

$$g_a = -\lambda_{m+a} + \frac{\partial L}{\partial u_a}. \tag{6.6}$$

Let M' denote the subspace of M obtained by setting

$$g_a = 0,$$

i.e.

$$\frac{\partial L}{\partial u_a} = \lambda_{m+a}. \tag{6.7}$$

Set:

$$\theta = \lambda_i dx_i + H\, dt \tag{6.8}$$

$$\theta' = \theta \text{ restricted to } M'. \tag{6.9}$$

We then have proved the following result:

<u>Theorem 6.1</u>. Suppose that $u_a(x, \lambda)$ are functions which satisfy relations (6.7). Set:

$$H'(x, \lambda) = H(x, u(x, \lambda), \lambda). \tag{6.10}$$

Then the extremals are the curves $t \to x(t), \lambda(t)$ which satisfy Hamilton's equations with Hamiltonian H'. In other words, they are the characteristic curves of the following closed 2-form on M':

$$d(\lambda_i dx_i + H'dt). \tag{6.11}$$

Now, let us rewrite the extremal equations in a more familiar "Lagrangian" form. To this end, let us write down the Hamilton equations explicitly, using formula (6.5).

We see from (6.7) that $u_a(x, \lambda)$ is a function of $(x, \lambda_{m+1}, \ldots, \lambda_{2m})$. In other words,

$$\frac{\partial u_a}{\partial \lambda_b} = 0.$$

Thus,

$$\frac{\partial H'}{\partial \lambda_{m+a}} = \frac{\partial L}{\partial u_b}\frac{\partial u_b}{\partial \lambda_{m+a}} - u_a - \lambda_{m+b}\frac{\partial u_a}{\partial \lambda_{m+a}}$$

$$= \text{, using (6.7),}$$

$$\frac{\partial H'}{\partial x_a} = \frac{\partial L}{\partial x_a} + \frac{\partial L}{\partial u_b}\frac{\partial u_b}{\partial x_a} - \lambda_{m+b}\frac{\partial u_b}{\partial x_a}$$

$$= \frac{\partial L}{\partial x_a}\text{, using (6.7).}$$

$$\frac{\partial H'}{\partial x_{m+a}} = \frac{\partial L}{\partial x_{m+a}} - \lambda_a + \frac{\partial L}{\partial u_b}\frac{\partial u_b}{\partial x_{m+a}}$$

$$- \lambda_{m+b}\frac{\partial u_a}{\partial x_{m+a}}$$

$$= \frac{\partial L}{\partial x_{m+a}} - \lambda_a\text{, using (6.7) again.}$$

We can then write down the Hamilton equations for Hamiltonian H', as follows:

$$\frac{dx_a}{dt} = -\frac{\partial H'}{\partial \lambda_a} = x_{m+a} \tag{6.12}$$

$$\frac{dx_{m+a}}{dt} = -\frac{\partial H'}{\partial \lambda_{m+a}} = u_a \tag{6.13}$$

$$\frac{d\lambda_a}{dt} = \frac{\partial H'}{\partial x_a} = \frac{\partial L}{\partial x_a} \tag{6.14}$$

$$\frac{d\lambda_{m+a}}{dt} = \frac{\partial H'}{\partial x_{m+a}} = \frac{\partial L}{\partial x_{m+a}} - \lambda_a. \tag{6.15}$$

Combining (6.12) and (6.13), we have:

$$\frac{d^2x_a}{dt^2} = u_a(x(t), \lambda(t)). \tag{6.16}$$

Now, using (6.16),

$$\begin{aligned}\lambda_{m+a}(t) &= \frac{\partial L}{\partial u_a} (x(t), u(x(t), \lambda(t)) \\ &= \frac{\partial L}{\partial u_a} (x(t), \frac{d^2x}{dt})\end{aligned} \tag{6.17}$$

Combine (6.15) and (6.17),

$$\begin{aligned}\frac{d}{dt} (\frac{\partial L}{\partial u_a} (x(t), \frac{d^2x}{dt^2}) = \frac{\partial L}{\partial x_{m+a}} (x(t), \frac{d^2x}{dt^2}) \\ - \lambda_a(t).\end{aligned} \tag{6.18}$$

To eliminate λ_a from (6.18), differentiate both sides with respect to t; and use (6.14)

$$\begin{aligned}&\frac{d^2}{dt^2} (\frac{\partial L}{\partial u_a} (x(t), \frac{d^2x}{dt^2})) \\ &\quad - \frac{d}{dt} (\frac{\partial L}{\partial x_{m+a}} (x(t), \frac{d^2x}{dt^2})) \\ &\quad + \frac{\partial L}{\partial x_a} (x(t), \frac{d^2x}{dt^2}) = 0.\end{aligned} \tag{6.19}$$

We can now transform back to the notation with which we began

$$x_a \rightarrow q_a$$

$$x_{m+a} \rightarrow \dot{q}_a$$

$$u_a \rightarrow \ddot{q}_a.$$

Equation (6.19) takes the following form:

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial u_a}\left(q(t), \frac{dq}{dt}, \frac{d^2q}{dt^2}\right)\right)$$
$$- \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\left(q(t), \frac{dq}{dt}, \frac{d^2q}{dt^2}\right)\right), \qquad (6.20)$$
$$+ \frac{\partial L}{\partial q_a}\left(q(t), \frac{dq}{dt}, \frac{d^2q}{dt^2}\right) = 0.$$

One recognizes that (6.20) are the classical Euler-Lagrange equations for the extremization of the variational problem:

$$\int L\left(q(t), \frac{dq}{dt}, \frac{d^2q}{dt^2}\right)dt.$$

We have then shown that the extremals defined by the optimal control-Hamiltons equation route are also Euler-Lagrange extremals. This then provides a check of the consistency of the formalism developed in previous sections.

<u>Exercise</u>. Does the converse hold, i.e. does every solution of (6.20) arise in this way from a solution of the Hamilton equations (6.12)-(6.15).

<u>Remark</u>: The Hamiltonian form, (6.12)-(6.15), of the Euler-Lagrange equations should be useful for the purpose of quantization of the equations.

<u>Problem</u>. Discuss the "quantum mechanics" of Lagrangians $L(q, \dot{q}, \ddot{q})$ that are polynomials of degree at most <u>two</u> in the variables $(q, \dot{q}, \ddot{q})$.

Chapter VI

SINGULAR CHARACTERISTIC CURVES OF CLOSED 2-FORMS AND OPTIMAL CONTROL VARIATIONAL PROBLEMS

1. INTRODUCTION

It is known, at least since E. Cartan's book "Leçons sur les invariants intégraux," that from a very broad differential geometric point of view, the calculus of variations is closely related to the study of the characteristic curves and integral manifolds of closed 2-forms. In DGCV and my paper "Some differential geometric aspects of the Lagrange variational problem " I have developed this point of view further, emphasizing the Lagrange variational problem (which Cartan did not treat.) As we have already seen, the optimal control variational problems, which ARE a subset of the "Lagrange" ones, are not "regular." In this chapter, we begin the development of that part of the general theory of closed 2-forms which will be useful in understanding the "singular" nature of the optimal control problems.

Let us begin by recalling some elementary differential-geometric ideas.

2. CHARACTERISTIC VECTORS AND CURVES OF CLOSED 2-FORMS

For the moment, we emphasize general differential geometric ideas. More specific situations closely related to optimal control theory will be considered later.

Let M be a manifold. Let ω be a closed 2-form on M. Let p be a point of M, and let M_p denote the tangent space to M at p.

<u>Definition</u>. A tangent vector $v \in M_p$ is a <u>characteristic vector</u> of ω if:

$$v \lrcorner \omega = 0, \tag{2.1}$$

i.e. if

$$\omega(v, M_p) = 0.$$

$C(p, \omega)$ denotes the space of characteristic vectors to ω at p.

Let t denote a real variable, ranging over some fixed interval; used, typically, to parameterize curves in M.

If $t \to \sigma(t)$ is a curve in M, $\sigma'(t)$ denotes the tangent vector to σ at t. It is an element of $M_{\sigma(t)}$, defined by the following formula:

$$\sigma'(t)(f) = \frac{d}{dt} f(\sigma(t)) \tag{2.2}$$

for all $f \in F(M)$.

Definition. A curve $t \to \sigma(t)$ in M is a characteristic curve for ω if:

$$\sigma'(t) \in C(\sigma(t), \omega) \tag{2.3}$$

for all t.

Another way of putting condition (2.3) is:

$$\sigma'(t) \lrcorner \omega = 0 \tag{2.4}$$

for all t.

Theorem 2.1. For $p \in M$,

$$\dim (C(q, \omega)) \leq \dim C(p, \omega) \tag{2.5}$$

for all points $q \in M$ which are sufficiently close to p.

Proof. For $p \in M$, let $M_p{}^d$ denote the dual space to M_p, i.e. the space of 1-covectors to M at p. Define a linear map

$$\alpha_p \colon M_p \to M_p{}^d$$

by the following formula:

$$\alpha_p(v) = v \lrcorner \omega$$

for $v \in M_p$.

Notice that:

$$C(p, \omega) = \text{kernel } \alpha_p.$$

Hence,

$$\dim M = \dim C(p, \omega) + \dim (\alpha_p(M_p)). \tag{2.6}$$

(This is the standard formula from linear algebra: rank + nullity of a linear map = dimension of domain.)

<u>to prove</u>:

$$\dim \alpha_q(M_q) \geq \dim \alpha_p(M_p) \tag{2.7}$$

for all $q \in M$ which are sufficiently close to p.

Pick elements $v_1, \ldots, v_r$ such that

$$\alpha_p(v_1), \ldots, \alpha_p(v_r)$$

are linearly independent elements of M_p^d, with:

$$r = \dim (\alpha_p(M_p)).$$

This means that the following 1-covectors

$$v_1 \lrcorner\, \omega, \ldots, v_r \lrcorner\, \omega$$

are linearly independent.

Let $X_1, \ldots, X_r$ be vector fields on M such that: $X_1(p) = v_1, \ldots, X_r(p) = v_r$. Then, by continuity

$$X_1(q) \lrcorner\, \omega, \ldots, X_r(q) \lrcorner\, \omega$$

are linearly independent for q sufficiently close to p. This proves (2.7). (2.5) now follows from (2.7) and (2.6).

<u>Definition</u>. A point $p \in M$ is a <u>non-singular point</u> for ω if:

$$\dim (C(q, \omega)) = \dim (C(p, \omega)) \tag{2.8}$$

for all q sufficiently close to p.

Let M' denote the set of non-singular points for ω. We see from its definition that M' is an open subset of M. The assignment

$$p \to C(p, \omega)$$

defines a constant-dimension vector field system on M'. It is known (e.g. see DGCV, Chapter 13) that this system is completely integrable. This defines what is sometimes called a <u>symplectic foliation</u> of M'. The leaves are the union of the non-singular characteristic curves passing through a given point, and the quotient space of M' by this manifold has (at least if it is a manifold) a <u>canonical structure</u>, i.e. a closed 2-form with no non-zero characteristic vectors, whose pull-back to M' is precisely ω. (See my paper, "Some differential-geometric aspects of the Lagrange variational problem").

However, in this chapter we are mainly interested in the singular characteristic vectors and curves. Now, there does not seem to be very much known <u>in general</u> about them. We have already encountered important examples of them in studying the variational problems of optimal control theory.

Accordingly, we shall continue working in this direction.

3. SINGULAR CHARACTERISTIC CURVES AND HAMILTONIAN SYSTEMS

Let us first review the relation between non-singular characteristic curves and Hamiltonian systems, familiar from classical mechanics (See E. Cartan's "Leçons sur les invariants integraux," and DGCV.)

Suppose that:

$$M = R^{2n+1}.$$

Label the Cartesian coordinates for M as follows:

$$(x_i, \lambda_i, t),$$

$$1 \le i, j \le n.$$

Let $H(x, \lambda, t)$ be a function of these variables, i.e. a real-valued function on M. Set:

$$\theta = \lambda_i dx_i + Hdt \tag{3.1}$$

$$\omega = d\theta. \tag{3.2}$$

$$= d\lambda_i \wedge dx_i + dH \wedge dt. \tag{3.3}$$

<u>Exercise</u>. Show that the dimension of the space of characteristic vectors of ω is one, at each point of M. In

particular, each characteristic curve of ω is non-singular. Show that, in fact, the characteristic curves are the curves $t \to (x(t), \lambda(y), t)$ in M, which satisfy the following Hamilton equations:

$$\begin{aligned} \frac{dx_i}{dt} &= -\frac{\partial H}{\partial \lambda_i} \\ \frac{d\lambda_i}{dt} &= \frac{\partial H}{\partial x_i} . \end{aligned} \tag{3.4}$$

These results establish the relation between the theory of characteristic curves and Hamilton's equations. Now, let us generalize to cover systems theory situations. Set

$$M = R^{2n+m+1} \tag{3.5}$$

Label the variables for M as follows:

$$\begin{gathered} (x_i, \lambda_i, u_a, t), \\ 1 \le i, j \le n \\ 1 \le a, b \le m. \end{gathered} \tag{3.6}$$

Let $H'(x, \lambda, u, t)$ be a function of these variables. Set:

$$\theta = \lambda_i dx_i + H'dt \tag{3.7}$$

$$\omega = d\theta. \tag{3.8}$$

Let us now compute the characteristic vectors of ω. Suppose given $v \in T(M)$. Then,

$$
\begin{aligned}
v \lrcorner \omega &= v \lrcorner d\lambda_i \wedge dx_i + dH \wedge dt \\
&= v(\lambda_i)dx_i - v(x_i)d\lambda_i \\
&+ v(H)dt - v(t)dH \\
&= (v(\lambda_i) - v(t) \frac{\partial H}{\partial x_i})dx_i \\
&- (v(x_i) + v(t) \frac{\partial H}{\partial \lambda_i})d\lambda_i \\
&+ (v(H) - v(t) \frac{\partial H}{\partial t})dt - v(t) \frac{\partial H}{\partial u_a} du_a
\end{aligned}
\tag{3.9}
$$

We can read off from this formula what the characteristic vectors of ω must be. Let p be a point of M. Set:

$V(p)$ = set of vectors $v \in M_p$ such that:

$$d\lambda_i(v) = 0 = dx_i(v) = dt(v) = dH(v). \tag{3.10}$$

Notice from (3.9) that:

$$V(p) \subset C(p, \omega). \tag{3.11}$$

Let us calculate the dimension of $V(p)$.

Suppose that $v \in V(p)$. Then, v is determined by the components:

$$du_a(v) = v_a. \tag{3.12}$$

(3.10) requires that:

$$\frac{\partial H}{\partial u_a} (p)v_a = 0. \tag{3.13}$$

Now, let M' be the subset of points $p \in M$ such that:

$$\frac{\partial H}{\partial u_a}(p) = 0. \tag{3.14}$$

(In other words, M' is the set of points at which H, as a function of u alone, with the other variables held fixed, has a critical point.)

<u>Theorem 3.1</u>. If $p \notin M'$, then

$$V(p) = C(p, \omega), \tag{3.15}$$

and

$$\dim V(p) = \dim C(p, \omega) = m-1. \tag{3.16}$$

<u>Proof</u>. $p \notin M'$ means that not all of the $(\frac{\partial H}{\partial u_a}(p))$ can be zero. If $v \in V(p)$, then v is determined by its components (v_a), $1 \leq a \leq m$ which satisfy the non-trivial linear relation (3.13). This proves (3.16).

Let us prove (3.15). Suppose $v \in C(p, \omega)$, i.e.

$$v \lrcorner \omega = 0.$$

In (3.9), the coefficients of du_a must vanish, i.e.

$$v(t) \frac{\partial H}{\partial u_a}(p) = 0.$$

Since not all the $(\frac{\partial H}{\partial u_a}(p))$ vanish, this forces $v(t) = 0$, which implies that $v \in V(p)$, and implies (3.15).

Theorem 3.2. If $p \in M'$, then:

$$V(p) \neq C(p, \omega) \tag{3.17}$$

$$\dim V(p) = M \tag{3.18}$$

$$\dim C(p, \omega) = m+1. \tag{3.19}$$

Proof. Using (3.9), we see that there is a unique $v \in C(p, \omega)$ such that

$$v(t) = 1; \; v(u_a) = 0.$$

In fact, we can read off from (3.9) the following formula for such v:

$$v = \frac{\partial}{\partial t} + \frac{\partial H}{\partial x_i}(p) \frac{\partial}{\partial \lambda_i} - \frac{\partial H}{\partial \lambda_i}(p) \frac{\partial}{\partial x_i}. \tag{3.20}$$

We see also that if $v \in V(p)$, the components $v_a \equiv v(u_a)$ can be arbitrary, which proves (3.18). Hence, $C(p, \omega)$ is the direct sum of $V(p)$ and the one dimensional subspace spanned by the v given by (3.20). This proves (3.19). (3.17) now follows also.

Remark. Theorems 3.1 and 3.2 now show precisely how the dimension of $C(p, \omega)$ varies as p ranges over M. In particular, the points of M' are singular, while the points of M-M' are regular.

4. BEHAVIOR OF CHARACTERISTIC CURVES UNDER MAPPINGS

Let us now pause in the development of the relation between characteristic curves and optimal control problems to survey some general facts about how characteristic curves behave under mappings.

Let M and M' be manifolds, and let ω be a closed 2-form on M, ω' a closed 2-form on M'. Let

$$\varphi: M' \to M$$

be a mapping such that:

$$\varphi^*(\omega) = \omega'. \tag{4.1}$$

In this section, we are interested in detailing how φ interrelates characteristic curves of ω and ω'.

<u>Theorem 4.1</u>. Suppose that $t \to \sigma(t)$ is a curve in M which is characteristic for ω, and that $t \to \sigma_1(t)$ is a curve in M' which is mapped into σ by φ. Then, σ_1 is also a characteristic curve of ω'.

<u>Proof</u>. We are given that:

$$\sigma(t) = \varphi(\sigma_1(t)) \text{ for all } t. \tag{4.2}$$

For $v' \in M'_{\sigma_1(t)}$,

$\omega'(\sigma_1{}'(t), v') =$, using (4.1),

$$\varphi^*(\omega)(\sigma_1'(t), v')$$
$$= \omega(\varphi_*(\sigma_1'(t)), \varphi_*(v'))$$
$$=, \text{ using (4.2)}, \omega(\sigma'(t), \varphi_*(v')),$$

which is zero because $t \to \sigma(t)$ is a characteristic curve of ω, i.e. $\sigma'(t) \lrcorner \omega = 0$.

<u>Theorem 4.2</u>. Suppose that φ is a maximal rank mapping of M onto M', and that $t \to \sigma_1(t)$ is a curve in M' which is characteristic for ω'. Let $t \to \sigma(t) = \varphi(\sigma_1(t))$ be the image curve in M. Then, $\sigma(t)$ is characteristic for ω.

<u>Proof</u>. The "maximal rank" property of φ means that:

$$\varphi_*(M_p') = M_{\varphi(p)} \tag{4.3}$$

for all $p \in M'$.

Hence,

$$\omega(\sigma'(t), M_{\sigma(t)}) = \omega(\varphi_*(\sigma_1'(t), \varphi_*(M_{\sigma_1(t)}))$$
$$= \varphi^*(\omega)(\sigma_1'(t), M_{\sigma_1(t)})$$
$$= \omega'(\sigma_1'(t), M_{\sigma_1(t)}),$$

which is zero because $t \to \sigma_1(t)$ is given as a characteristic curve of ω'.

Theorems 4.1 and 4.2 are almost trivial, of course.

However, we shall find them useful in studying optimal control problems.

5. REGULARIZATION OF SINGULAR CHARACTERISTIC CURVES

Let us now formulate another simple, general idea. (We shall see that it provides a conceptual framework for a situation to be encountered in control theory problems.)

Let M be a manifold, with a closed 2-form ω given on M. Let $t \to \sigma(t)$ be a singular characteristic curvw for ω.

Definition. A quadruple $(M', \varphi, \omega', \sigma_1)$ consisting of a manifold M', a map $\varphi: M' \to M$, a closed 2-form ω' on M', and a characteristic curve σ_1 of ω' is said to be a regularization of the singular characteristic curve σ if the following conditions are satisfied:

$$\varphi(\sigma_1(t)) = \sigma(t) \tag{5.1}$$

for all t

$t \to \sigma_1(t)$ is a non-singular characteristic curve of ω'. (5.2)

$$\varphi^*(\omega) = \omega'. \tag{5.3}$$

This is the "regularization" concept, considered for a single singular curve σ. One may also formulate the regularization concept, as applied to all singular characteristic curves.

Definition. A triple (M', φ, ω') consisting of a manifold M', a map $\varphi: M' \to M$ and a closed 2-form ω' on M' is said to be a regularization of the singular characteristic curves of ω if the following conditions are satisfied:

$$\varphi^*(\omega) = \omega' \tag{5.4}$$

ω' is non-singular at each point of M'. (5.5)

For each singular characteristic curve $t \to \sigma(t)$ of ω, there is a characteristic curve $t \to \sigma_1(t)$ of ω' such that $\varphi(\sigma_1(t)) = \sigma(t)$ for all t. (5.6)

We shall see in the next section that there is an important example of such a regularization arising from control theory.

Remark: This "regularization" concept is a variant of the general idea of "prolongation of a singular solution of an exterior differential system to be a non-singular solution," developed by E. Cartan. There is a discussion of this idea in my book "Geometry, physics and systems," as well as the original discussion in Cartan's book "Les systèmes extérieures et leurs applications géométriques" (which is notoriously hard to understand!)

6. REGULARIZATION OF A SINGULAR CLOSED 2-FORM ARISING IN CONTROL THEORY

Return now to the special situation discussed in Section 3.

$$M = R^{2n+m+1}. \tag{6.1}$$

The variables on M are labelled

$$(x_i, \lambda_i, t, u_a), \quad 1 \le i, j \le n;\ 1 \le a, b \le m \tag{6.2}$$

$H(x, \lambda, t, u)$ is a function on M. In this section we shall suppose that the following condition is satisfied.

$$\det\left(\frac{\partial^2 H}{\partial u_a \partial u_b}\right) \ne 0 \tag{6.3}$$

at each point of M.

<u>Remark</u>: Condition (6.3) is analogous to what is called in the classical calculus of variations a "regularity condition." Our use of this term here would obviously lead to confusion,

and we shall not do so.

With this data given, let us set:

$$\theta = \lambda_i dx_i + H\,dt \tag{6.4}$$

$$y_a = \frac{\partial H}{\partial u_a} \tag{6.5}$$

$$\omega = d\theta \tag{6.6}$$

Thus, θ is a 1-form, y_a functions, and ω a closed 2-form, on M. Define M' as follows:

$$\begin{aligned} M' = {} & \text{set of points of } M \text{ at which} \\ & \text{the functions } y_a \text{ vanish.} \end{aligned} \tag{6.7}$$

<u>Theorem 6.1</u>. M' is a regularly embedded submanifold of M of dimension 2n+1.

<u>Proof</u>. Let us first show that the differentials dy_a are linearly independent at each point of M, i.e. the functions f_a are functionally independent.

Now,

$$\begin{aligned} dy_a = {} & \frac{\partial^2 H}{\partial u_b \partial u_a}\, du_b \\ & + \frac{\partial^2 H}{\partial x_i \partial u_a}\, dx_i + \frac{\partial^2 H}{\partial \lambda_i \partial u_a}\, d\lambda_i \\ & + \frac{\partial^2 H'}{\partial t \partial u_a}\, dt. \end{aligned} \tag{6.8}$$

If the dy_a were dependent, there would be non-zero real numbers c_a such that:

$$c_a dy_a = 0. \tag{6.9}$$

(6.8) and (6.9) combine to force the condition:

$$c_a \frac{\partial^2 H}{\partial u_b \partial u_b} = 0. \tag{6.10}$$

(6.3) and (6.10) combine to force

$$c_a = 0, \text{ contradiction.}$$

<u>Exercise</u>. Show (using the implicit function theorem, as described, for example, in DGCV) that the independence of the dy_a implies the statement of Theorem 6.1.

Let

$$\varphi: M' \to M$$

be the inclusion mapping. Set

$$\omega' = \varphi^*(\omega).$$

$$x_i' = \varphi^*(x_i)$$

$$\lambda_i' = \varphi^*(x_i)$$

$$t' = \varphi^*(t)$$

$$H' = \varphi^*(H)$$

$$u_a' = \varphi^*(u_a).$$

<u>Theorem 6.2</u>. The functions (x_i', λ_i', t) form a coordinate system for M'.

<u>Proof</u>. By Theorem 6.1, we know that M' is $(2n+1)$-dimensional. Hence, to prove Theorem 6.2 we must show that the forms dx_i', $d\lambda_i'$, dt' are linearly independent.

Otherwise, there would be relations of the following form:

$$b_i dx_i' + c_i d\lambda_i' + b dt' = 0$$

or

$$b_i dx_i + c_i d\lambda_i' + b dt = e_a dy_a \tag{6.11}$$

Using relations (6.8) again, we see that:

$$c_a \frac{\partial^2 H}{\partial u_a \partial u_b} = 0.$$

(6.3) now forces

$$c_a = 0. \tag{6.12}$$

Insert (6.12) in (6.11). The resulting dependency relation among the dx_i, $d\lambda_i$, dt must be trivial, i.e.

$$b_i = c_i = b = 0,$$

which finishes the proof.

Using Theorem 6.2, we see that the u_a' are functions of the (x', λ', t). Hence, H' is of the following form:

$$H'(x', \lambda', t') = H(x', \lambda', t', u'(x', \lambda', t')). \tag{6.13}$$

Hence,

$$\omega' = d(\lambda', dx' + H'dt'). \tag{6.14}$$

Now, we shall take the "prime" off the variables, hoping that the reader can keep the distinction in his mind. Putting all these results together proves the following main result:

Theorem 6.3. Suppose that $t \to (x(t), \lambda(t), t, u(t))$ is a curve in M. This curve is then a singular characteristic curve of ω if and only if the following Hamiltonian differential equations are satisfied:

$$\begin{aligned} \frac{dx_i}{dt} &= -\frac{\partial H'}{\partial \lambda_i}(x(t), \lambda(t), t) \\ \frac{d\lambda_i}{dt} &= \frac{\partial H'}{\partial x_i}(x(t), \lambda(t), t). \end{aligned} \tag{6.15}$$

Proof. We know that such a curve is singular if and only if it lies on M'. If it lies on M', it is a characteristic curve for ω'. But, formula (6.14) describes the "Hamiltonian" form of ω', and it is known (see DGCV) that the characteristic curves of 2-form of type (6.14) are solutions of the Hamilton equations.

Remark: Let us emphasize again the key qualitative idea in the facts described above. ω is originally given as a singular closed 2-form on a manifold M. A submanifold M' of M is found so that ω restricted to M' is a non-singular closed 2-form ω' of maximal rank, and so that the singular characteristic curves of ω are given also as the characteristic curves of ω'.

Chapter VII

THE ALGEBRA OF BOUNDARY CONDITIONS FOR LINEAR ORDINARY DIFFERENTIAL EQUATIONS

1. INTRODUCTION

In this chapter I will describe some of the algebraic ideas associated with the study of boundary value problems for ordinary differential equations. As references for the classical material in this area see Ince [1] and Hille [1]. Here, we shall emphasize the role of general algebraic and geometric ideas in describing the classical theory, and its generalizations. In particular, I will relate the general theory of Lie systems of ordinary differential equations (see Chapter I) to the boundary value problems. One can refer to Volume III of this treatise for a treatment of the theory of linear Lie systems and of auxillary algebraic and geometric ideas which will also play a role in this chapter.

2. THE GRASSMAN SPACE AND AFFINE SUBSPACES

K will denote the real or complex numbers. V will be a finite dimensional vector space, with K as its field of scalars. L(V) denotes the space of K-linear maps: $V \to V$.

Suppose that V is an n dimensional vector space. Let m be an any integer between 1 and n. An m-dimensional linear subspace of V is a linear subspace of V which is m-dimensional in its own right as a K-vector space.

G(V) is the space of all linear subspaces of V. A typical element of G(V) is denoted by γ. G(V) is called the Grassman space of V. $G^m(V)$ denotes the subset of G(V) consisting of those linear subspaces which are m-dimensional. Recall (from Volume III, for example) that the study of G(V) plays a key role in the theory of matrix Riccati equations and generalized linear fractional transformations. We shall see in this chapter that it also underlies the theory of boundary value problems.

Here is a key idea.

Definition. Let γ_0, γ_1 be two elements of G(V), of dimension m_0 and m_1, respectively. Then, in case one of the following conditions are satisfied, γ_0 and γ_1 are said to be in general position:

a) $\gamma_0 \cap \gamma_1 = (0)$ and $m_0 + m_1 \le \dim V$. (2.1)

b) $\dim(\gamma_0 \cap \gamma_1) = m_0 + m_1 - \dim V$ and $m_0 + m_1 \ge \dim V$. (2.2)

Exercise. Prove the following basic geometric property of the "general position" notion.

> If γ_0, γ_1 are in general position, and if
> $\gamma_0' \in G^{m_1}(V)$ is sufficiently close to γ_0,
> $\gamma_1' \in G^{m_1}(V)$ is sufficiently close to γ_1, (2.3)
> then γ_0', γ_1' are in general position.

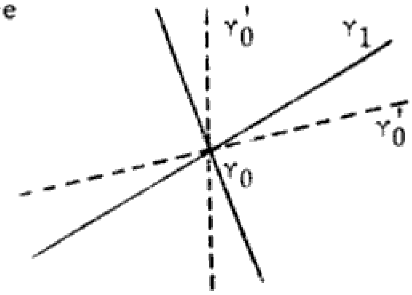
In other words, the set of all elements in $G^{m_0}(V) \times G^{m_1}(V)$ which are in general position forms an open subset, with respect to the natural topology on these Grassman spaces.

Remark: The basic intuitive idea of (2.3) is easy to understand, even if the reader does not understand the fancy jargon. The elements of $\gamma_0 \cap \gamma_1$ are determined as solutions of a set of linear equations. The "General Position" condition means basically that the rank of the matrix determining these linear equations is maximal. Thus, if one changes γ_0 and γ_1 slightly, the matrix changes slightly, and its rank remains maximal.

Example. These notions are most intuitive in the case $V = R^3$, because they are then classical geometric notions. Suppose first:

Case 1. $\mathbf{m}_0 = 1 = \mathbf{m}_1$

Elements of $G^1(V)$ are then lines in R^3 through the origin. "General position" means that they intersect only at the origin. See figure



Case 2. $\mathfrak{m}_0 = 2, \mathfrak{m}_1 = 1$.

Elements of $G^2(V)$ are planes through the origin. "General Position" in this case means that a plane and a line meet only at the origin.

Case 3. $\mathfrak{m}_0 = 2 = \mathfrak{m}_1$.

Then, $\mathfrak{m}_0 + \mathfrak{m}_2 - \dim V = 1$.

Elements of $G^2(V)$ are planes. Two such planes are in general position if they meet in a line.

Now let us study the "affine" generalization of these ideas.

Definition. A subset a of V is an affine subspace of V if the following subset of V is a linear subspace

$$\{v_1 - v_2 : v_1, v_2 \in a\}. \tag{2.4}$$

The dimension of the vector space (2.4) is also called the dimension of a.

Let α be such an affine subspace, and let γ be the linear subspace satisfying (2.4).

Exercise. Generalize the "general position" notion to affine subspaces, and show that two affine subspaces which are in general position can be perturbed slightly and remain in general position.

3. BOUNDARY CONDITIONS FOR LINEAR DIFFERENTIAL EQUATIONS

Let V continue to be a finite dimensional real or complex vector space. L(V) denotes the linear maps: $V \to V$. Let t be a real parameter, $0 \le t \le \infty$. Let

$$t \to A(t) \in L(V)$$

be a one parameter family of linear maps, i.e. a curve in L(V). Suppose A(t) depends continuously on t. This curve defines a linear differential equation:

$$\frac{dv}{dt} = A(t)v. \tag{3.1}$$

A solution of (3.1) is a curve $t \to v(t)$ in V whose derivative $\frac{dv}{dt}$ satisfies (3.1).

Let GL(V) denote the group of invertible linear maps

$$g: V \to V.$$

A linear flow on V is defined as a differentiable curve

$$t \to g(t)$$

in GL(V) such that:

$$g(t) = 1.$$

A curve $t \to v(t)$ in V is an orbit of the flow if there is a vector $v_0 \in V$ such that:

$$v(t) = g(t)(v_0) \tag{3.2}$$

for $0 \le t < \infty$.

Let us find the differential equation satisfied by the orbits:

$$\frac{dv}{dt} = \frac{dg}{dt}(v_0) = \frac{dg}{dt} g(t)^{-1}(v(t)).$$

Each orbit is then a solution of (3.1) if and only if:

$$\frac{dg}{dt} g(t)^{-1} = A(t) \tag{3.3}$$

for $0 \le t < \infty$.

Conversely, the differential equation (3.1) determines the flow, after solving the operator differential equation:

$$\frac{dg}{dt} = A(t)g. \tag{3.4}$$

The connection between linear flows and linear differential equations is discussed more extensively in Volume III. Of course, the general theory is of differential equations of type (3.1) is discussed extensively in the standard treatises on ordinary differential equations, e.g. Gantmacher [1], Coddington and Levinson [1], Hartman [1] and Hille [1]. We will call $A(t)$ the _infinitesimal generator of the flow_. In turn, this linear theory is a special case of more general equations on manifolds which are called _Lie systems_ in the classical literature, and which were studied by Lie himself and by E. Vessiot.

We shall now study the differential equations (3.1) themselves and their boundary conditions. First, we must modify (3.1) slightly to allow "inhomogeneous" equations.

Definition. An _inhomogeneous differential equation_ associated to (3.1) is one of the following form:

$$\frac{dv}{dt} = A(t)\, v + u(t), \tag{3.5}$$

where $t \to u(t)$ is a given curve in V. Of course, the well-known "variation of parameters" formula enables one to write the general solution of (3.5) in terms of the flow $t \to g(t)$ which is the solution of (3.4).

Definition. A _boundary condition_ is a pair (γ_0, γ_1) of elements of $G(V)$ such that:

$$\dim \gamma_0 + \dim \gamma_1 = \dim V. \tag{3.6}$$

A curve $t \to v(t)$, $0 \le t \le a$, in V is said to satisfy _the_

<u>boundary condition</u> if:

$$v(0) \in \gamma_0, \ v(a) \in \gamma_1. \tag{3.7}$$

<u>Remark</u>: a is a real number satisfying

$$0 < a < \infty.$$

Typically, in mathematical physics we are interested in finding solutions of an inhomogeneous differential equation of type (3.5), which <u>also</u> satisfy the boundary conditions (3.7).

Often, one is interested strongly in uniqueness of the solution to a boundary value problem. It can be posed in the following way.

Suppose given $v_0 \in \gamma_0$, $v_1 \in \gamma_1$. Suppose we want to find a solution $t \to v(t)$, $0 \le t \le a$, of (3.5) which satisfies (3.7). Suppose two of them exist. Their difference, $t \to w(t)$, is a curve in V, which satisfies the differential equation (3.1), and the boundary condition (3.7). Let $t \to g(t)$ be the flow in GL(V) whose infinitesimal generator is $t \to A(t)$. Thus,

$$w(t) = g(t)(w(0)).$$

If $w(0) \in \gamma_0$, $w(a) \in \gamma_1$, we then have:

$$w(a) \in g(a)(\gamma_0) \cap \gamma_1.$$

<u>If</u> $g(a)(\gamma_0)$ and γ_1 are in general position, as defined in

Section 3, then $w(a) = 0$, which forces (by uniqueness of solutions of (3.1))

$$w(t) \equiv 0.$$

We can sum up as follows:

Theorem 3.1. If $g(a)(\gamma_0)$ and γ_1 are in general position, the solutions of the boundary value problem defined above are unique.

Of course, this result is largely just a restatement of the definition. But it is often useful because it is "geometric," and there are non-trivial qualitative geometric techniques available for proving things in this way.

4. SYMPLECTIC FLOWS AND SELF-ADJOINT BOUNDARY CONDITIONS

In our work on "matrix Riccati equations" in volume III we have seen that the "symplectic flows" and "Lagrangian decompositions" of V play a specially important role in relating the general ideas to important practical situations. Simiiliarly, in our study of the mathematics of "boundary conditions," we shall show that the same mathematical ideas are also important.

In addition to the data given in previous sections, suppose that ω is given as a skew-symmetric, non-degenerate bilinear form

$$\omega\colon V \times V \to K.$$

For each t, suppose that:

$$\omega(A(t)v_1, v_2) + \omega(v_1, A(t)v_2) = 0 \tag{4.1}$$

for $v_1, v_2 \in V$.

(4.1) means that $t \to A(t)$ is a infinitesimal symplectic automorphism. It is the condition that the flow $t \to g(t)$ defined by (3.4) is an automorphism of ω.

A Lagrangian subspace of V is a linear subspace Q of V such that

$$\omega(Q, Q) = 0. \tag{4.2}$$

A Lagrangian decomposition of V is a pair (Q, P) of Lagrangian subspaces such that

$$V = Q \oplus P. \tag{4.3}$$

Let us suppose that such a decomposition is fixed. Then, A(t) determines linear maps:

$$\begin{aligned} &B(t)\colon Q \to Q, \; C(t)\colon Q \to P \\ &D(t)\colon P \to Q, \; E(t)\colon P \to P \end{aligned} \tag{4.4}$$

such that:

$$\begin{aligned} A(t)(q) &= B(t)q + C(t)q \\ A(t)(p) &= D(t)p + E(t)p \end{aligned} \tag{4.5}$$

for $p \in P$, $q \in Q$.

Condition (4.1) imposes the following conditions on these maps:

$$\begin{aligned} 0 &= \omega(A(t)q_1, q_2) + \omega(q_1, A(t)q_2) \\ &= \omega(Bq_1 + Cq_1, q_2) + \omega(q_1, Bq_2 + Cq_2) \\ &= \omega(Cq_1, q_2) + \omega(q_1, Cq_2). \end{aligned} \tag{4.6}$$

$$\begin{aligned} 0 &= \omega(Aq, p) + \omega(q, Ap) \\ &= \omega(Bq + Cq, p) + \omega(q, Dp + Ep) \\ &= \omega(Bq, p) + \omega(q, Ep). \end{aligned} \tag{4.7}$$

$$0 = \omega(Dp_1, p_2) + \omega(p_1, Dp_2) \tag{4.8}$$

for $q, q_1, q_2 \in Q, p, p_1, p_2 \in P$.

Suppose also that:

$$\dim V = 2n. \tag{4.9}$$

Definition. A set $(\gamma_0, \gamma_1) \in G(V) \times G(V)$ of boundary conditions is said to be self-adjoint if the following conditions are satisfied:

$$\dim \gamma_0 = \dim \gamma_1 = n \tag{4.10}$$

γ_0, γ_1 are symplectic subspaces of V. (4.11)

There are maps $X_0, X_1: Q \to P$ such that:

$$\gamma_0 = \{q + X_0(q): q \in Q\}$$

$$\gamma_1 = \{q + X_1(q): q \in Q\}.$$

Let us write down the conditions that (4.11) imposes on the maps X_0, X_1:

$$\begin{aligned} 0 &= \omega(q_1 + X_0(q_1), q_2 + X_0(q_2)) \\ &= \omega(q_1, X_0(q_2)) + \omega(X_0q_1, q_2) \end{aligned} \tag{4.12}$$

$$0 = \omega(q_1, X_1, q_2) + \omega(X_1 q_1, q_2) \tag{4.13}$$
for $q_1, q_2 \in Q$.

Remarks: A linear subspace of V which satisfies (4.10) and (4.11) is said to be maximal Lagrangian. The space of elements of G(V) which are maximal Lagrangian is a manifold M. In fact it is an orbit of Sp(n, K), the group of symplectic automorphisms.

The set of all elements of M which can be written as $\{q + X(q): q \in Q\}$, for some map $X: Q \to P$, forms an open subset M' of M. If a basis (q_i) for Q, (p_i) for P, is chosen so that:

$$\omega(q_i, p_j) = \delta_{ij}, \quad 1 \le i, j \le n,$$

then condition (4.12) means that

$$X(q_i) = X_{ji} p_j,$$

where (X_{ij}) is a symmetric matrix. This identifies M' with the set of $m \times m$ symmetric matrices, with coefficients in K. These spaces are important in the theory of automorphic functions.

Suppose we consider a differential equation, with a boundary condition:

$$\frac{dv}{dt} = A(t)v. \tag{4.14}$$

$$v(0) \in \gamma_0, \quad v(a) \in \gamma_1. \tag{4.15}$$

Suppose

$$v(t) = q(t) \oplus p(t), \tag{4.16}$$

with $q(t) \in Q$, $p(t) \in P$. Then, system (4.14) is equivalent to the following pair of equations for $t \to q(t)$, $p(t)$:

$$\begin{aligned} \frac{dq}{dt} &= B(t)q + D(t)p \\ \frac{dp}{dt} &= C(t)q + E(t)p. \end{aligned} \tag{4.17}$$

Relations (4.6)-(4.8) describe how the linear map B, C, D, E are related to each other and to ω.

Suppose now that γ_0, γ_1 are self-adjoint boundary conditions, determined by linear maps

$$X_0, X_1 \colon Q \to P$$

which satisfy (4.12) and (4.13). The boundary conditions (4.14)-(4.15) then take the following form:

$$p(0) = X_0(q(0)). \tag{4.18}$$

$$p(1) = X_1(q(1)). \tag{4.19}$$

These equations may be specialized to a more traditional matrix form. Suppose

$$P = Q = K^n. \tag{4.20}$$

Thus, a point $p \in P$ or $q \in Q$ is n-vector $(p_1, \ldots, p_n)$ or $(q_1, \ldots, q_n)$. Suppose also that is defined as follows:

$$\omega(q, p) = q_1 p_1 + \ldots + q_n p_n. \tag{4.21}$$

Then, $B(t)$, $C(t)$, $D(t)$, $E(t)$ are determined by $n \times n$ matrices, for which we use the same letter. Let $B(t)^T, \ldots$ denote the transposes of these matrices.

Exercise. Show that conditions (4.6)-(4.8) are equivalent to the following matrix conditions:

$$C(t)^T = C(t) \tag{4.22}$$

$$D(t)^T = D(t) \tag{4.23}$$

$$B(t)^T = -E(t). \tag{4.24}$$

Equations (4.17) may now be specialized to give many familiar forms. For example, suppose that:

$$D(t)^{-1} \text{ exists.} \tag{4.25}$$

Then,

$$p = D^{-1} \frac{dq}{dt} - D^{-1}Bq.$$

Hence,

$$\frac{d}{dt} p(t) = Cq + Ep = Cq + E(D^{-1} \frac{dq}{dt} - D^{-1}Bq)$$

or

$$\frac{d}{dt} (D^{-1} \frac{dq}{dt}) = Cq - ED^{-1}Bq + ED^{-1} \frac{dq}{dt} + \frac{d}{dt} (D^{-1}Bq).$$

Use (4.24), and write the equations in the following form, which is equivalent to (4.17):

$$\frac{d}{dt} (D^{-1} \frac{dq}{dt}) = \frac{d}{dt} (D^{-1}Bq) - B^TD^{-1} \frac{dq}{dt} + (C + B^TD^{-1}B)q. \tag{4.26}$$

Similiarly, the boundary conditions (4.18)-(4.19) may be written as follows:

$$D(0)^{-1} \frac{dq}{dt}(0) - D(0)^{-1}B(0)q(0) = X_0 q(0) \quad (4.27)$$

$$D(a)^{-1} \frac{dq}{dt}(a) - D(a)^{-1}B(a)q(a) = X_1 q(1). \quad (4.28)$$

Conditions (4.12)-(4.13) mean that X_0, X_1 are $n \times n$ symmetric matrices.

We can now explain where the term "self-adjoint" come from. Suppose, for simplicity, that the ground field K is R, and that:

$$D(t) = 1 \text{ for all } t. \quad (4.29)$$

Let H be the space of C^∞ curves $t \to q(t)$ in $R^n = Q$, $0 \le t \le a$, satisfying the boundary conditions (4.27)-(4.28). Define the symmetric inner product $< \, , \, >$ on H as follows:

$$<q_1, q_2> = \int_0^a q_1(t) \cdot q_2(t) dt \quad (4.30)$$

($\cdot$ denotes the usual dot product in R^n). Let α be the linear differential operator

$$\alpha: H \to H$$

defined as follows:

$$\alpha(q) = \frac{d^2q}{dt^2} - \frac{d}{dt}(Bq) + B^T \frac{dq}{dt} + (C + B^TB)q \quad (4.31)$$

Theorem 4.1. α is a symmetric differential operator, i.e.

$$\langle \alpha q_1, q_2 \rangle = \langle q_1, \alpha q_2 \rangle \tag{4.32}$$

for $q_1, q_2 \in H$.

Proof. By (4.21), C is a symmetric matrix so is $B^T B$. Hence, the third term on the right hand side of (4.31) is obviously a symmetric operator. We must then only consider the first two terms. The boundary conditions satisfied by elements $t \to q(t)$ of H take the following form:

$$\frac{dq}{dt}(0) = B(0)q(0) + X_0 q(0) \tag{4.33}$$

$$\frac{dq}{dt}(a) = B(a)q(a) + X_1(q(a)). \tag{4.34}$$

Then,

$$\int_0^a q_1(t)\cdot\left(\frac{d^2 q_2}{dt^2} - \frac{d}{dt} Bq_2 + B^T \frac{dq_2}{dt}\right)dt$$

$$= q_1(t)\cdot \frac{dq_2}{dt}\Big|_{t=0}^{t=a} - \int_0^a \frac{dq_1}{dt}\cdot\frac{dq_2}{dt}\,dt$$

$$- q_1(t)\cdot B(t)q_2(t)\Big|_0^a + \int_0^a \frac{dq_1}{dt}\cdot B(t)q_2 dt$$

$$+ \int_0^a Bq_1 \cdot \frac{dq_2}{dt}\,dt$$

=, using (4.33)-(4.34),

$$
\begin{aligned}
& q_1(a)\cdot(B(a)q_2(a) + X_1q_2(a)) \\
& - q_1(0)\cdot(B(0)q_2(0) + X_0q_2(0)) \\
& - q_1(a)\cdot B(a)q_2(a) + q_1(0)\cdot B(0)q_2(0) \\
& - \frac{dq_1}{dt}\cdot q_2\Big|_0^a + \int_0^a \frac{d^2q_1}{dt^2}\cdot q_2\,dt \\
& + \int_0^a B(t)^T\frac{dq_1}{dt}\cdot q_2 + B(q_1)\cdot q_2\Big|_0^a \\
& - \int_0^a \frac{d}{dt}Bq_1\cdot q_2\,dt \\
& = -(B(a)q_1(a)\cdot q_2(a) + X_1q_1(a)\cdot q_2(a)) + q_1(a)\cdot X_1q_2(a) \\
& - q_1(0)\cdot X_0q_2(0) + B(0)q_1(0)\cdot q_2(0) + X_0q_1(0)\cdot q_2(0) \\
& + \int_0^a \left(\frac{d^2q_1}{dt^2} + B(t)^T\frac{dq_1}{dt} - \frac{d}{dt}Bq_1\right)\cdot q_2\,dt \\
& + B(a)q_1(a)\cdot q_2(0) - B(0)q_1(0)\cdot q_2(0).
\end{aligned}
$$

Now comes the crucial moment when all the terms must cancel, except the integral, in order to prove Theorem 4.1. The non-trivial question is whether the terms involving X_0 and X_1 cancel. Notice that they do indeed cancel, but symmetry of X_0 and X_1 is required for this. In turn, symmetry of X_0 and X_1 is equivalent to the condition that the subspaces γ_0, γ_1 of V defined by (4.11) be maximal

Lagrangian. This explains why we chose to give the name "self-adjoint" to the boundary conditions determined by maximal Lagrangian γ_0, γ_1.

<u>Remark</u>. In systems theory, "reciprocal constitutive relations" are also determined by maximal Lagrangian subspaces of a vector space with a symplectic form. Presumably, this mathematical identity between "reciprocal constitutive relations" and "self-adjoint boundary conditions" is no accident, but reflects an interesting and important general feature of physical theories.

5. SECOND ORDER LINEAR SCALAR EQUATIONS IN HAMILTONIAN FORM

We have seen that the algebra of "boundary values" takes on a very elegant algebraic form when a given system of ordinary differential equations is written in first order form. If this first order form is Hamiltonian, we can then formulate algebraically what is meant by a "self-adjoint" boundary value problem.

Thus, a key question is often how to write a given system in Hamiltonian form. This is also an important question for quantum mechanics, because the most convenient starting point for the process of "quantization" of a differential equation (or of the underlying physical system) is to write it in Hamiltonian form.

Now, if one starts with differential equations which arise from a variational principle, there is a well-known process for writing it in Hamiltonian form. Conversely, Hamiltonian systems arise from variational principles. Therefore, it would be interesting to have available a direct analysis of the conditions that a system is equivalent to one in Hamiltonian form. I know of no systematic method for doing this, which works in adequate generality. Therefore, I will simply work with the simplest non-trivial case, a second order linear ordinary differential equation for one unknown function, say of the form:

$$\frac{d^2x}{dt} = a(t)\,\frac{dx}{dt} + b(t)x, \tag{5.1}$$

$t \to a(t)$, $b(t)$ are given real valued functions of the real variable t. "x" denotes a real variable, and (5.1) is to be solved for a function $t \to x(t)$. The work in this section will be very elementary, but it will be useful to have on hand as a source of intuition about more complicated situations.

Let us now explain what is meant by "writing (5.1) in Hamiltonian form." Introduce another real variable y. Let H(x,y,t) be a function of these variables. Write down the Hamilton equations, with Hamiltonian H:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y} \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x} \end{aligned} \tag{5.2}$$

What is required is that for every solution $t \to (x(t), y(t))$ of (5.2), the projected curve $t \to x(t)$ be a solution of (5.1), and that conversely every solution of (5.1) arise in this way by projecting a solution of (5.2).

We shall restrict attention to H's which lead to a linear set of equations (5.2), i.e. which are of the following form:

$$H = \frac{1}{2}\,\alpha(t)x^2 + \beta(t)xy + \frac{1}{2}\,\gamma(t)y^2. \tag{5.3}$$

<u>Problem</u>. Are there more general possibilities for H? Then, the equations (5.2) take the following form:

$$\begin{aligned} \frac{dx}{dt} &= \beta(t)x + \gamma(t)y \\ \frac{dy}{dt} &= -\alpha(t)x - \beta(t)y \end{aligned} \tag{5.4}$$

We must find the conditions that the projected curve be a solution of (5.1).

$$\gamma(t) \neq 0 \text{ for all } t \neq 0. \tag{5.5}$$

Now, differentiating (5.4), we have:

$$\begin{aligned} \frac{d^2x}{dt} &= \frac{d\beta}{dt}\,x + \beta\,\frac{dx}{dt} + \frac{d\gamma}{dt}\,y + \gamma\,\frac{dy}{dt} \\ &= \frac{d\beta}{dt}\,x + \beta\,\frac{dx}{dt} + \frac{d\gamma}{dt}\,y - \gamma(\alpha x + \beta y) \end{aligned}$$

$$= x(\frac{d\beta}{dt} - \alpha\gamma) + \beta \frac{dx}{dt} + y(\frac{d\gamma}{dt} - \gamma\beta)$$

$$= x(\frac{d\beta}{dt} - \alpha\gamma) + \beta \frac{dx}{dt} + \frac{1}{\gamma} (\frac{dx}{dt} - \beta x)(\frac{d\gamma}{dt} - \gamma\beta). \qquad (5.6)$$

Now, $x(0)$ and $y(0)$ can be prescribed arbitrarily. (5.5) has the consequence that $x(0)$ and $\frac{dx}{dt}(0)$ can also take on arbitrary values; thus, in order that every solution of (5.4) project into a solution of (5.1), we must have equality between the coefficients of x and of $\frac{dx}{dt}$ on the right hand side of (5.1) and (5.6), i.e.

$$\beta + \frac{1}{\gamma} (\frac{d}{dt} - \gamma\beta) = a$$

or

$$\frac{1}{\gamma} \frac{d\gamma}{dt} = a \qquad (5.7)$$

$$\frac{d\beta}{dt} - \alpha\gamma - \frac{\beta}{\gamma} (\frac{d\gamma}{dt} - \gamma\beta) = b$$

or, using (5.7),

$$\frac{d\beta}{dt} - \alpha\beta - a\beta + \beta^2 = b. \qquad (5.8)$$

Let us sum up as follows:

<u>Theorem 5.1</u>. Suppose $\gamma(t) \neq 0$ for all t, and that every solution of (5.4) projects into a solution of (5.1). Then, every solution of (5.1) arises as a projection of exactly

one solution of (5.4), and coefficients (a,b) are determined in terms of (α,β,γ) by Equations (5.7) and (5.8).

Proof. If $\gamma(t) \neq 0$, note that the correspondence $(x(t_0), y(t_0)) \to (x(t_0), \frac{dx}{dt}(t_0))$ between initial conditions of both equations is determined by an invertible linear equation, which proves that the solutions correspond.

Remark: Given (a,b), one can determine (α,β,γ) by solving for them, i.e. by solving the differential equations (5.7) and (5.8). In particular, notice that (5.8) is a Ri catti equation for β, so it cannot always be solved by elementary methods. Presumably there are interesting problems when one asks for global solutions, with perhaps certain types of asymptotic behavior.

Now, let us suppose that:

$$\gamma(t) = 0 \text{ for all } t. \tag{5.9}$$

Then, (5.4) implies the following "constraint" on the curve $t \to x(t)$:

$$\frac{dx}{dt} = \beta x. \tag{5.10}$$

Differentiate (5.10):

$$\frac{d^2x}{dt^2} = \frac{d\beta}{dt} x + \beta \frac{dx}{dt}$$

$$= (\frac{d}{dt} + \beta^2)x$$

$=$, if $t \to x(t)$ is a solution of (5.1),

$a\beta x + bx.$

Thus, if $x \neq 0$, we must have:

$$a\beta + b = \frac{d\beta}{dt} + \beta^2. \tag{5.11}$$

Again, this is a Riccati equation for β. α can obviously be chosen arbitrarily.

Exercise. Analyze what can happen if:

$$\gamma(0) = 0;\ \gamma(t) \neq 0 \text{ for } t \neq 0. \tag{5.12}$$

Exercise. Show that the condition $\gamma(t) \equiv 0$ means that the flow generated by Equation (5.4) remains within a subgroup of $Sp(1, R)$. Determine this subgroup, and prove that it is a solvable group.

Chapter VIII

THE KRONECKER THEORY OF SYSTEMS OF LINEAR, TIME-INVARIANT ORDINARY DIFFERENTIAL EQUATIONS

1. INTRODUCTION

Let V, W be real vector spaces, and let

$$\alpha_0, \alpha_1: V \to W$$

be linear maps. Construct the following linear differential equation:

$$\alpha_1 \frac{dv}{dt} = -\alpha_0 v, \tag{1.1}$$

to be solved for a curve

$$t \to v(t)$$

Much of what is called, in the engineering literature, "linear system theory" is concerned with algebraic properties of solutions of (1.1). (The "analytic" properties are usually easy to study, once the relevant "algebra" is isolated.) However, the systems studied in applications usually have a special form, which enables one to proceed without a general "algebraic" theory. For example, the system might be of "input-output" form, as studied in Volume VIII. The recent book by Rosenbrock [1] studies rather general systems, but he does not directly use a systematic algebraic theory.

Gantmacher's treatise on matrix theory provides (in Chapter 12) an exposition of a general algebraic "structure" theory (due to Kronecker) of systems of type (1.1). In this chapter, I will redo this theory, from a more contemporary algebraic point of view. My aim is not only expository, but to prepare the way to understanding a possible "structure" theory which would apply to non-linear and partial differential equation systems.

The algebra really involves the theory of modules over rings of polynomials. I shall then change notations slightly from those used in Volume III in order to conform to algebro-geometric ideas introduced in Volume VIII. Let us now turn to a description of these notations and ideas.

2. THE WEIRSTRASS-KRONECKER-SMITH THEORY

The classical Weirstrass "elementary divisor" theory (developed expertly by Gantmacher [1]) deals with normal forms of matrices with polynomial coefficients. As I understand the historical situation by reading Gantmacher, Kronecker developed certain special features which play an important role in the study of equations (1.1). The recent books an systems theory (e.g. Rosenbrock [1]) name after Smith the "normal form" that Gantmacher credits to Weirstrass. I am not familiar with the historical situation here, and therefore give the name of all three men to the material in this section.

Let V, W be finite dimensional vector spaces, with the complex numbers C as field of scalars. (In fact, everything can be generalized to arbitrary fields of scalars). In Volume VIII I have explained in general what is meant by a "polynomial map." They typically occur in two versions, as linear combinations of "indeterminates," and as true mappings. The whole point of the theory is to look at them in both ways.

Here are some basic notations:

C = field of complex numbers.

$C[\lambda]$ = ring of polynomials in an "indeterminate" λ.

$C[\lambda]$ also is P(C, C), the polynomial maps: $C \to C$.

$V[\lambda]$ = linear combinations of powers of λ, with coefficients in V,

$= \{v_0 + \lambda v_1 + \dots + \lambda^n v_n : v_0, \dots, v_n \in V\}$

$V[\lambda]$ also may be considered as the space.

$P(C, V) \equiv$ polynomial maps: $C \to V$.

$V[\lambda]$ is a $C[\lambda]$-module. A "vector polynomial"

$$\underset{\sim}{v} = v_0 + \lambda v_1 + \ldots + \lambda^n v_n$$

is multiplied by a scalar polynomial

$$p = c_0 + c_1\lambda + \ldots + c_m\lambda^m$$

as one would expect:

$$p\underset{\sim}{v} = c_0 v_0 + (c_1 v_0 + c_0 v)\lambda + \ldots + c_m v_n \lambda^{m+n}$$

Let

$$L(V, W)$$

denote the vector space of linear maps: $V \to W$. Thus,

$$L(V, W)[\lambda]$$

denotes the λ-polynomials, with coefficients in λ, or, alternately,

$$L(V, W)[\lambda] = P(\lambda, L(V, W))$$

$$\equiv \text{polynomial maps } C \to L(V, W).$$

<u>Definition</u>. Suppose $n = \dim V$. A set $(v_1, \ldots, v_n)$ of elements of $V[\lambda]$ is said to form a <u>Weirstrass basis</u> of the $C[\lambda]$-module $V[\lambda]$ is there are elements

$$(\underset{\sim}{\omega}_1, \ldots, \underset{\sim}{\omega}_n) \in V^d[\lambda]$$

such that:

$$\wp_i(\chi_j) = \delta_{ij} \tag{2.1}$$

for $1 \le i, j \le n$.

Let us see what is involved in this definition. Suppose

(v_i) is

a basis of V. Then, there are relations of the following form:

$$\chi_i = f_{ij}v_j, \tag{2.2}$$

with $f_{ij} \in C[\lambda]$. (The summation convention is in force). Suppose that (ω_i) is the dual basis of V^d, i.e.

$$\omega_i(v_j) = \delta_{ij}. \tag{2.3}$$

<u>Suppose</u> that $\wp_i$ satisfying (2.1) exist. Then,

$$\wp_i = f_{ij}'\omega_j.$$

Hence,

$$\begin{aligned} \delta_{ij} &= \wp_i(\chi_j) \\ &= f_{ik}'\omega_k(f_{j\ell}v_\ell) \\ &= f_{ik}'f_{j\ell}\delta_{k\ell} \\ &= f_{ik}'f_{jk}. \end{aligned} \tag{2.4}$$

Conversely, <u>if</u> equations (2.4) may be solved for (f_{ik}'), then the $\wp_i$ satisfying (2.1) exist, i.e. $(\chi_1, \ldots, \chi_n)$ is a Kronecker basis of $V[\lambda]$. Notice further that equations (2.4) say that the matrix (f_{ik}') is the transpose of the inverse of the matrix (f_{ij}). Cramer's rule (which gives a formula for the inverse, involving polynomials in the matrix elements as numerator and the determinant in the denominator) gives a formula for f_{ij}', hence gives a sufficient condition that χ_i define a Weirstrass basis, namely that:

$$\text{determinant } (f_{ij}) \text{ is a non-zero constant.} \tag{2.5}$$

We can then sum up as follows:

Theorem 2.1. Suppose that $(\chi_1', \ldots, \chi_n')$ is a given Kronecker basis for $V[\lambda]$. (For example, a basis consisting of constant vectors, as considered above). Let (f_{ij}) be an $n \times n$ matrix of scalar polynomials, and let:

$$\chi_i = f_{ij}\chi_j'. \tag{2.6}$$

Suppose that (2.5) is satisfied. Then, $(\chi_1, \ldots, \chi_n)$ is a Kronecker basis of $V[\lambda]$.

Exercise. Give an example (e.g. in the case $n = 2$) where (2.5) is not necessary for $\chi_1, \ldots, \chi_n$ to form a Kronecker basis.

The following definition formalizes another aspect of this concept.

Definition. Let $(v_1, \ldots, v_n)$, $(v_1', \ldots, v_n')$ be two bases of the module $V[\lambda]$. Suppose that:

$$v_i' = f_{ij} v_j, \tag{2.7}$$

where (f_{ij}) is an $n \times n$ matrix of scalar polynomials. The two bases are said to be Weirstrass equivalent if the determinant of the (f_{ij}) is a non-zero constant.

Remarks: Theorem 2.1 can be restated as follows: If (v_i') is a Weirstrass basis, and if (v_i) is a module basis which is Weirstrass-equivalent to the basis (v_i'), then (v_i) is also a Weirstrass basis.

Further, one readily sees that the notion of "Weirstrass equivalence" is really an equivalence relation. These equivalence relations play a key, but hidden, relation in system theory.

We have now developed the algebraic concepts needed to state the main Weirstrass-Kronecker-Smith result in suitably general form.

Theorem 2.2. Let V, W be finite dimensional vector spaces, and let g be an element of $L(V, W)[\lambda]$. Suppose that:

$$\dim V = n$$
$$\dim W = m.$$

Let $(\underset{\sim}{v}_1', \ldots, \underset{\sim}{v}_n')$, $(\underset{\sim}{w}_1', \ldots, \underset{\sim}{w}_m')$ be given bases of $V[\lambda]$ and $W[\lambda]$. Then, there is an integer r, with

$$r \le m, n,$$

new bases

$$(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n), \; (\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_m)$$

which are Kronecker equivalent to the old bases, and non-zero polynomials

$$p_1, \ldots, p_r \in C[\lambda]$$

such that:

$$\begin{aligned} \underset{\sim}{a}(\underset{\sim}{v}_1) &= p_1\underset{\sim}{w}_1 \\ \underset{\sim}{a}(\underset{\sim}{v}_2) &= (p_1p_2)\underset{\sim}{w}_2 \\ &\vdots \\ \underset{\sim}{a}(\underset{\sim}{v}_r) &= (p_1\cdots p_r)\underset{\sim}{w}_r \\ \underset{\sim}{a}(\underset{\sim}{v}_i) &= 0 \text{ for } r+1 \le i \le n. \end{aligned} \tag{2.8}$$

The polynomials $(p_1, \ldots, p_r)$ are called the <u>elementary factors</u> of $\underset{\sim}{a}$.

<u>Proof</u>. Choose indices as follows, and the summation convention:

$$1 \le i, j \le n$$
$$1 \le a, b \le m.$$

Suppose (f_{ai}) is the $m \times n$ matrix of polynomials such that:

$$\underset{\sim}{g}(\underset{\sim}{v}_i') = f_{ai}\underset{\sim}{w}_a' \tag{2.9}$$

We now subject the matrix

$$\begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \\ f_{m1} & \cdots & f_{mn} \end{pmatrix}$$

to the sort of "elementary" row and column transformations developed in the classical theory, e.g. in Gantmacher's treatise, Chapter 6. I have already sketched how this would go, in the module language, in Chapter 3, Section 3, Volume III. The key observation is that such an elementary row or column transformation corresponds to a choice of basis of $V[\lambda]$ or $W[\lambda]$ <u>within the same Weirstrass equivalence class</u>. For completeness, I go over this argument again.

Suppose the initial bases of $V[\lambda]$, $W[\lambda]$ are chosen so that, among <u>all</u> matrix representations (f_{ai}) of $\underset{\sim}{g}$, f_{11} is of minimal non-zero degree, say s. Then, each f_{1i} can be divided by f_{11}, with a remainder, i.e.

$$f_{1i} = q_i f_{11} + r_i, \ i \geq 2, \tag{2.10}$$

with:

$$q_i, r_i \in C[\lambda],$$

$$\text{degree } r_i < s.$$

Set:

$$v_1'' = v_1'; \; v_i'' = v_i' - q_i v_1' \text{ for } i > 1. \quad (2.11)$$

Then,

$$g(v_1'') = \alpha(v_1') = f_{a1} w_a'. \quad (2.12)$$

For $i \geq 2$,

$$\begin{aligned} g(v_i'') &= g(v_i') - q_i g(v_1') \\ &= f_{ai} w_a' - q_i f_{a1} w_a' \\ &=, \text{ using (2.10)}, \end{aligned}$$

$$r_i w_1' + (\text{terms in } w_2', \ldots, w_m').$$

By the minimal way we have chosen s,

$$r_i = 0,$$

since otherwise polynomials of degree less than s would appear in a matrix realization of g. Notice that the change of bases (2.11) from the prime system to the double prime one is a Weirstrass equivalence, since the determinant of the matrix of polynomials effecting the change is 1. In this new basis, the matrix (f_{ai}') representing g satisfies:

$$f_{1i}' = 0 \text{ for } i \geq 1,$$

i.e. has zeros in the first row:

$$\begin{pmatrix} f_{11}, & 0,\ldots, & 0 \\ * & * \ldots & x \\ x & x \ldots & x \end{pmatrix}$$

In a similiar way the basis of $W[\lambda]$ can be changed so that there are zeros in the first column:

$$\begin{pmatrix} f_{11}, & 0_1,\ldots, & 0 \\ 0 & * & * \\ \vdots & & \\ 0 & * & * \end{pmatrix}$$

Now, suppose the elements of the bases are permuted so that:

$$f_{22} \neq 0.$$

(If this is not possible, all matrix elements are zero, and $\underline{g}$ is already in its canonical form (2.8), with $r = 0$.) Again, divide f_{22} by f_{11}, with a remainder ρ of degree less than s. If ρ were non-zero, a Weirstrass change of basis can be made making it a matrix element, contradiction, and forming it to be zero. Thus,

$$f_{22} \text{ is a multiple (in } C[\lambda]) \text{ of } f_{11}.$$

Set:

$$p_1 = f_{11}$$

$$f_{22} = p_1 p_2.$$

Again, choose the ordering of the basis so that the degree of f_{22} is minimal. One can then make changes of basis of the form (2.11) to make the off-diagonal elements of the second row and column equal to zero. Continuing in this way, one ends up with canonical form (2.8), and the proof of Theorem 2.2.

<u>Exercise and Remark</u>. Suppose $(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n)$, $(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_m)$, $(\underset{\sim}{v}_1', \ldots, \underset{\sim}{v}_n')$, $(\underset{\sim}{w}_1', \ldots, \underset{\sim}{w}_m')$ are two Weirstrass-equivalence sets of basis in which $\underset{\sim}{g}$ takes the canonical form (2.8), with polynomials $(p_1, \ldots, p_r)$, $(p_1', \ldots, p_{r'}')$ whose highest coefficient is 1. Show that:

$$r = r'$$

$$p_1 = p_1', \ldots, p_r = p_r'.$$

This result is extremely important from the "invariant theory" point of view. Let: $GL(n, C[\lambda], W)$ denote the set of all $n \times n$ matrices, with matrix elements in $C[\lambda]$, whose determinant is a non-zero constant. $GL(n, C[\lambda], W)$ forms a group, which one may call the <u>Weirstrass group</u>. Let $M(n, m, C[\lambda])$ denote the set of $n \times m$ matrices, with coefficients in $C[\lambda]$. Ordinary pre-matrix multiplication by $GL(n, C[\lambda], W)$ and post-multiplication by $GL(m, C[\lambda], W)$ defines an action of

$$GL(n, C[\lambda], W) \times GL(m, C[\lambda], W)$$

on $M(n, m, C[\lambda])$. The above Exercise then implies that $p_1, \ldots, p_r$ are <u>invariants</u> of the action of this group.

<u>Exercise</u>. Show (at least in simple cases, e.g. $n = m = 2$) how the invariants $p_1, \ldots, p_n$ may be calculated explicitly in terms of the determinants of the square matrix "minors" of the matrices in $M(n, m, C[\lambda])$.

These results all hold when C is replaced by an arbitrary field K. In fact, they hold when $C[\lambda]$ is replaced by a arbitrary principal ideal domain, and $V[\lambda]$, $W[\lambda]$ are replaced by <u>free</u> finite dimensional modules over this domain. The ultimate setting for these ideas is probably "algebraic K-theory."

In certain cases (see Gantmacher) the orbits of $GL(n, C[\lambda], W) \times GL(m, C[\lambda], W)$ on certain points of $M(n, m, C[\lambda])$ are <u>the same</u> as the orbits of the subgroup

$$GL(n, C) \times GL(m, C)$$

consisting of the <u>constant</u> matrices. For example, if:

$$\dim V = \dim M, \text{ i.e.}$$

$$m = n$$

$$\underset{\sim}{a} = a_0 + \lambda a_1,$$

with:

$$\det(\underset{\sim}{\alpha}(\lambda)) \neq 0 \text{ for } \underline{\text{same}}\ \lambda \in C.$$

In this case, $\underset{\sim}{\alpha}$ is called, classically, a <u>regular pencil</u>. (The term "pencil" refers to the fact that $\underset{\sim}{\alpha}$ is a first degree polynomial in λ. The "Kronecker theory" extends this "Weirstrass theory" in case that the pencil is not regular). An important special case of a "regular pencil" (important, e.g. for the "Jordon Normal Form" problem, or the "elementary divisors of an n × n matrix" concept) is that where:

$$\underset{\sim}{\alpha} = 1 - \lambda\alpha_1.$$

In summary, in this Section we have briefly described (in module-language) the classical material on "canonical forms" of matrices whose coefficients are polynomials in one variable λ. Let us now return to the study of differential equations.

3. ALGEBRAIC STRUCTURE OF LINEAR CONSTANT COEFFICIENT DIFFERENTIAL EQUATIONS

Let V, W continue as finite dimensional, complex vector spaces. Consider a differential equation of the following form:

$$\alpha_n \frac{d^n v}{dt^n} + \ldots + \alpha_0 v = 0, \tag{3.1}$$

to be solved for a curve

$$t \to v(t) \text{ in } V,$$

where

$$\alpha_n, \ldots, \alpha_0 \colon V \to W$$

are linear maps.

Set:

$$\mathfrak{g}(\lambda) = \alpha_n \lambda^n + \ldots + \alpha_0. \tag{3.2}$$

Thus, $\mathfrak{g}$ is an element of

$$L(V, W)[\lambda].$$

The system (3.1) can be written as:

$$\mathfrak{g}\left(\frac{d}{dt}\right)(v) = 0. \tag{3.3}$$

It will be denoted by

$$(V, W, \mathfrak{g}).$$

<u>Definition</u>. Given two systems

$$(V, W, \mathfrak{g}), \ (V', W', \mathfrak{g}'),$$

we say that the latter is a <u>subsystem of the former</u> if the following conditions are satisfied:

$$V' \subset V; \ W' \subset W \tag{3.4}$$

$$\mathfrak{g}(\lambda) \text{ restricted to } V' = \mathfrak{g}(\lambda)'.$$

If $(V', W', \mathfrak{g}')$ is a subsystem of $(V, W, \mathfrak{g})$, then one can form a third system $(V'', W'', \mathfrak{g}'')$ as follows:

$$V'' = V/V'$$

$$W'' = W/W'$$

$\mathfrak{g}''(\lambda)$ is the quotient map, i.e. is defined by the following commutative diagram:

$$\begin{tikzcd}
V' \arrow[r, "\mathfrak{g}(\lambda)'"] \arrow[d] & W' \arrow[d] \\
V \arrow[r, "\mathfrak{g}(\lambda)"] \arrow[d] & W \arrow[d] \\
V'' \arrow[r, "\mathfrak{g}(\lambda)''"] & W''
\end{tikzcd}$$

$(V'', W'', \mathfrak{g}'')$ is called the <u>quotient system</u> of $(V, W, \mathfrak{g})$ by $(V', W', \mathfrak{g}')$. It is denoted as follows:

$$(V'', W'', \mathfrak{g}'') = (V, W, \mathfrak{g})/(V', W', \mathfrak{g}').$$

Now, suppose that

$$(V', W', \mathfrak{g}'), \ (V'', W'', \mathfrak{g}'')$$

are two systems. Form a third system $(V, W, \mathfrak{g})$ as follows:

$$V = V' \oplus V''$$

$$W = W' \oplus W''$$

$$\mathfrak{g}(\lambda) = \mathfrak{g}(\lambda)' \oplus \mathfrak{g}(\lambda)''$$

$(V, W, \mathfrak{g})$ is called the <u>direct sum</u> of $(V', W', \mathfrak{g}')$ and $(V'', W'', \mathfrak{g}'')$.

Notations.

We write:

$$(V', W', \mathfrak{g}') \subset (V, W, \mathfrak{g})$$

if $(V', W', \mathfrak{g}')$ is a subsystem of $(V, W, \mathfrak{g})$.

$$(V'', W'', \mathfrak{g}'') = (V, W, \mathfrak{g})/(V', W', \mathfrak{g}')$$

if $(V'', W'', \mathfrak{g}'')$ is the quotient of $(V, W, \mathfrak{g})$ by $(V', W', \mathfrak{g}')$. $(V, W, \mathfrak{g}) = (V', W', \mathfrak{g}') \oplus (V'', W'', \mathfrak{g}'')$ if $(V, W, \mathfrak{g})$ is the direct sum of $(V', W', \mathfrak{a}')$ and $(V'', W'', \mathfrak{g}'')$.

Remark: If $(V, W, \mathfrak{g})$ is the direct sum of $(V', W', \mathfrak{g}')$ and $(V'', W'', \mathfrak{g}'')$, as defined above, then $(V', W', \mathfrak{g}')$ and $(V'', W'', \mathfrak{g}'')$ are subsystems of $(V, W, \mathfrak{g})$.

These definitions essentially define what is meant by the "algebraic structure" of systems of linear, constant coefficient ordinary differential equations, just as, say, the notions of "subgroup," "quotient group," and "direct sum group" define the notion of "algebraic structure" for group theory. This somewhat vague idea can be made precise by means of the ideas of category theory, but I do not want to get into that at this point.

For example, here are two further important notions:

Definition. A system $(V, W, \mathfrak{g})$ is irreducible if it cannot be exhibited as the direct sum of two non-zero subsystems.

A system $(V, W, \mathfrak{g})$ is completely reducible if it is the direct sum of irreducible subsystems.

We shall now pause in our development of the general theory to consider the special case:

$$\frac{dv}{dt} + \alpha_0 v = 0.$$

We shall see how the "Jordan normal form" theory (see Volume II) fits in with this general algebraic viewpoint.

4. THE JORDAN THEORY AS A SPECIAL CASE OF A SYSTEM STRUCTURE THEORY

Suppose the system $(V, W, \mathfrak{g})$ takes the following special form:

$$V = W \tag{4.1}$$

$$\mathfrak{g}(\lambda) = \lambda + \alpha_0, \tag{4.2}$$

where α_0 is a linear map

$$V \to V.$$

Theorem 4.1. Suppose there is a complex number λ_0 and a basis $(v_1, \dots, v_n)$ of V such that:

$$
\begin{aligned}
(\alpha_0 - \lambda_0)(v_1) &= v_2 \\
(\alpha_0 - \lambda_0)(v_2) &= v_3 \\
&\vdots \\
(\alpha_0 - \lambda_0)(v_{n-1}) &= v_n \\
(\alpha_0 - \lambda_0)(v_n) &= 0.
\end{aligned}
\tag{4.3}
$$

Then, the system defined by (4.1)-(4.2) is irreducible.

Proof. Suppose the system is not irreducible. Let

$$
\begin{aligned}
V &= V' \oplus V'' \\
V \equiv W &= W' \oplus W''
\end{aligned}
$$

be a reduction. Now,

$$\alpha_1 = 1 = \text{identity map.}$$

Hence, the condition that:

$$\underset{\sim}{g}(\lambda)(V') \subset W'$$

for all λ implies that

$$
\begin{aligned}
V' &\subset W' \\
V'' &\subset W''
\end{aligned}
$$

By a dimensionality argument,

$$
\begin{aligned}
V' &= W' \\
V'' &= W''.
\end{aligned}
$$

Hence, also:

$$\alpha_0(V') \subset V'$$

$$\alpha_0(V'') \subset V''.$$

Also,

$$(\alpha_0 - \lambda_0)(V') \subset V'$$

$$(\alpha_0 - \lambda_0)(V'') \subset V''.$$

Now, (4.3) means that:

$$(\alpha_0 - \lambda_0)^n = 0 \tag{4.4}$$

Further,

There is at least one
vector $v \in V$ (e.g. v_1) (4.5)
such that $(\alpha_0 - \lambda_0)^n(v) \neq 0$.

<u>Remark</u>: (4.4)-(4.5) mean that

$$(z - \lambda_0)^n$$

is the <u>minimal polynomial</u> of α_0.

(4.4) also implies that $(\alpha_0 - \lambda_0)$ is <u>nilpotent</u>. Hence, $(\alpha_0 - \lambda_0)$ acting in V', V'' are nilpotent.

<u>Exercise</u>. Set $n' = \dim V'$, $n'' = \dim V''$. Show that:

$$(\alpha_0 - \lambda_0)^{n'}(V') = 0. \tag{4.6}$$

$$(\alpha_0 - \lambda_0)^{n''}(V'') = 0. \tag{4.7}$$

(4.6), (4.7) and the condition

$$V = V' \oplus V''$$

now imply that

$$(\alpha_0 - \lambda_0) = 0, \tag{4.7}$$

where

$$m = \max(n', n'') < n,$$

which is the contradiction.

Exercise. Prove the converse of this, i.e. if $(V, V, \lambda + \alpha_0)$ is irreducible, then there is a $\lambda_0 \in C$ such that:

$$(\alpha_0 - \lambda_0)^n = 0,$$

where $n = \dim V$.

Theorem 4.2. The system $(V, V, \mathfrak{g}(\lambda) \equiv \lambda + \alpha_0)$ is completely reducible.

Exercise. Prove Theorem 4.2, using the "Jordan decomposition" of α_0, as described in Chapter II of Volume II.

5. SUFFICIENT CONDITIONS FOR COMPLETE REDUCIBILITY

Let V, W continue as vector spaces,

$$\lambda \to g(\lambda) \in L(V, W)[\lambda]$$

is a polynomial map which defines a system.

Let

$$V_1 \subset V$$

$$W_1 \subset W$$

be a pair of linear subspaces which define a subsystem, i.e.

$$g(\lambda)(V_1) \subset W_1. \tag{5.1}$$

Let

$$V_2 \subset V$$

$$W_2 \subset W$$

be arbitrary linear subspaces such that:

$$\begin{aligned} V &= V_1 \oplus V_2 \\ W &= W_1 \oplus W_2. \end{aligned} \tag{5.2}$$

Of course, we cannot expect that such a pair of subspaces, chosen at random subject only to satisfy (5.2), would also define a subsystem, hence also a complete reduction of g. Instead, we will start with the vector space direct sum decomposition (5.2), "perturb" it, and examine the conditions that the perturbed subspace reduce the system g.

To define these "perturbed" subspaces, let

$$\begin{aligned} A&: V_2 \to V_1 \\ B&: W_2 \to W_1 \end{aligned} \tag{5.3}$$

be arbitrary linear maps. Set:

$$\begin{aligned} V_2' &= \{v_2 + A(v_2): v_2 \in V_2\} \\ W_2' &= \{w_2 + B(w_2): w_2 \in W_2\}. \end{aligned} \tag{5.4}$$

Exercise. Show that:

$$\begin{aligned} V &= V_1 \oplus V_2' \\ W &= W_1 \oplus W_2'. \end{aligned} \tag{5.5}$$

Exercise. Show that if (V_2', W_2') are arbitrary linear subspaces of (V, W) which satisfy (5.5), they are defined via formula (5.4), by appropriate choice of linear map A, B.

Now, for each $\lambda \in C$, let

$$\alpha_2(\lambda): V_2 \to W_2 \tag{5.6}$$

be the linear map defined by the following formula:

$$\begin{aligned} &\alpha_2(\lambda)(v_2) = \text{projection of } \alpha(\lambda)(v_2) \\ &\text{on } W_2\text{, with the projection defined} \\ &\text{by the direct-sum decomposition (5.2)} \end{aligned} \tag{5.7}$$

As λ varies, this defines a system

$$(V_2, W_2, \mathcal{Q}_2).$$

In order to identify this system, recall that the quotient system of (V, W, g) with respect to the subsystem (V_1, W_1) is defined by passing $g(\lambda)$ to the quotient

$$V/V_1 \to W/W_1.$$

But, in view of the direct sum decomposition (5.2),

$$\begin{aligned} &v/V_1 \text{ is isomorphic to } V_2 \\ &W/W_1 \text{ is isomorphic to } W_2. \end{aligned} \tag{5.8}$$

Exercise. Show that formula (5.6), plus the isomorphisms (5.8), prove that the quotient system

$$(V, W, g)/(V_1, W_1, g_1)$$

is

$$(V_2, W_2, g_2).$$

Remark: g_1 is the map associated with the subsystem, i.e.

$$g_1(\lambda) = g(\lambda) \text{ restricted to } V.$$

With the aid of this quotient system, let us return to the problem of finding conditions the maps A and B must satisfy in order that (V_2', W_2') defined by (5.4) also define a subsystem.

Now, for $v_2 \in V_2$, suppose that

$$g(\lambda)(v_2 + A(v_2)) \in W_2.$$

This means that there exists $w_2 \in W_2$ such that:

$$\begin{aligned} &\mathcal{g}(\lambda)(v_2) + \mathcal{g}(\lambda)(A(v_2)) \\ &\qquad = w_2 + B(w_2). \end{aligned} \tag{5.9}$$

Keeping in mind the relations these maps must satisfy, we have

$$\mathcal{g}_2(\lambda)(v_2) = w_2 \tag{5.10}$$

$$\begin{aligned} &\text{projection of } (\mathcal{g}(\lambda)(v_2) \text{ on } W_1 \\ &\qquad = - \mathcal{g}(\lambda)(A(v_2)) + B(w_2) \end{aligned} \tag{5.11}$$

<u>Define</u> a map

$\mathcal{B}(\lambda)\colon V_2 \to W_1$ as follows:

$$\begin{aligned} &\mathcal{B}(\lambda)(v_2) = \text{projection of} \\ &\mathcal{g}(\lambda)(v_2) \text{ on } W_1. \end{aligned} \tag{5.12}$$

Thus, we can combine (5.10)-(5.12) as follows:

$$\begin{aligned} &B\mathcal{g}_2(\lambda) - \mathcal{g}_1(\lambda)A \\ &\qquad = \mathcal{B}(\lambda) \end{aligned} \tag{5.13}$$

for all $\lambda \in C$.

This is the relation we need. Let us sum up as follows:

<u>Theorem 5.1</u>. The existence of a pair (A, B) of linear maps (with domain and range defined by (5.3)) satisfying (5.12) is necessary and sufficient for the existence of a direct

sum decomposition of the system (V, W, g), with the subsystem (V_1, W_1, g_1) as one of the summands and with the quotient system

$$(V, W, g)/(V_1, W_1, g_1)$$

as the other summand.

Proof. Necessity has been proved above. To prove sufficiency, one should notice that the above argument can be run in the reverse direction.

Notice that (5.13) is a linear, inhomogeneous equation for the pair (A, B) of linear maps. In typical situations, one will not know very much about $g(\lambda)$. Hence it is desirable to find conditions that (5.13) be solvable for A, B no matter what the choice of g. In fact, we shall see that the success of the "Kronecker" structure theory lies in precisely such a condition.

Before getting into the general Kronecker theory, we shall pause to discuss some simple, and classical, examples.

6. DIRECT SUM DECOMPOSITIONS ASSOCIATED WITH ONE MAP

First, suppose that:

$$g(\lambda) = \alpha_0 : V \to W, \tag{6.1}$$

i.e. $\mathfrak{g}$ is a polynomial of degree zero. Set:

$$V_1 = \text{kernel of } \alpha_0 \equiv \{v \in V: \alpha_0(v) = 0\} \qquad (6.2)$$

Let V_2 be any linear subspace of V such that:

$$V = V_1 \oplus V_2. \qquad (6.3)$$

Then, because of (6.2),

α_0 is one-one on V_2.

$\alpha_0(V_2) = \alpha_0(V)$.

Then, $(V_1, 0)$ and (V_2, W) form subsystems of $(V, W, \mathfrak{g})$, and obviously $(V, W, \mathfrak{g})$ is a direct sum of these subsystems. The subsystems $(V_1, 0)$ and (V_2, W) can then be split into irreducible subsystems with one-dimensional images and domains by choosing bases for V_1 and V_2.

Remarks: This decomposition of the system into the direct sum of irreducible subsystems corresponds to the classical (rather trivial) result in matrix theory that a arbitrary $m \times n$ matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

can be written in "canonical form" as follows, by appropriate choice of invertible $m \times m$ and $n \times n$ matrices M_1, M_2:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = M_1 \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & 1 & & & & \\ \vdots & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots \\ 0 & & & & & 0 \end{pmatrix} M_2$$

Now we can deal with a similiar "systems theory" framework for the <u>Jordan canonical form</u>. Let:

$$V = W \tag{6.4}$$

$$g(\lambda) = \lambda + \alpha_0, \tag{6.5}$$

where α_0 is a linear map: $V \to V$. We have already seen, in Section 4, that the "Jordan canonical form" for α_0 defines a decomposition of (V, V, g) as a direct sum of irreducible subsystems. Let us look at this, from the point of view of Section 5.

As explained in Section 4, the decomposition problem can be reduced to the case where α_0 is nilpotent, i.e.

$$\alpha_0{}^n = 0 \text{ for } n \text{ sufficiently large.} \tag{6.6}$$

Let n be the smallest integer satisfying (6.6). It is called the <u>index of nilpotency</u>. Here is the main result.

<u>Theorem 6.1</u>. Let

$$\alpha_0 \colon V \to V$$

be a nilpotent linear map, with n its index of nilpotency. Let

$$v_0 \in V$$

be an element such that

$$\alpha_0^{n-1}(v_0) \neq 0,$$

and let V_1 be the linear subspace of V spanned by $v_0, \alpha_0(v_0), \ldots \alpha_0^{n-1}(v_0)$. Then, there is a linear subspace

$$V_2 \subset V$$

such that:

$$V = V_1 \oplus V_2 \tag{6.7}$$

$$\alpha_0(V_2) \subset V_2. \tag{6.8}$$

In other words, the subsystem defined by V_1 is part of a direct sum decomposition.

The direct proof of Theorem 6.1 is given by Halmos [1], p. 109. It is the main result needed for the proof of the Jordan Canonical Form.

Exercise. Can Theorem 6.1 be proven by using the technique developed in Section 5?

Exercise. Is the conclusion of Theorem 6.1 true if v_0 is an element such that

$$\alpha_0^{n-1}(v_0) = 0?$$

if not, try to explain intuitively why it breaks down.

After this diversion to understand the reducibility theory for a single linear map, let us return to the Kronecker theory.

7. THE KRONECKER THEORY OF SINGULAR SYSTEMS

V, W are finite dimensional vector spaces,

$$\alpha_0, \alpha_1 : V \to W$$

are linear maps, and

$$\underset{\sim}{g}(\lambda) = \alpha_0 + \lambda\alpha_1 \tag{7.1}$$

(V, W, $\underset{\sim}{g}$) is called a Kronecker system.

Remark. I gave it this name because it was Kronecker (at least according to Gantmacher [1]) who first developed the complete algebraic structure theory for these systems. The "input-output systems" of the type considered by engineers are special cases of Kronecker systems.

Definition. The Kronecker system (V, W, $\underset{\sim}{g}$) is singular if the following condition is satisfied:

$$\underline{a}(\lambda) \text{ is not one-one} \tag{7.2}$$

for all $\lambda \in C$.

Recall that

$$V[\lambda]$$

denotes the set of polynomials

$$\underline{v}(\lambda) = v_0 + \lambda v_1 + \lambda^2 v_2 + \ldots + \lambda^n v_n,$$

with coefficients in V. The subspace of those of degree n is denoted by

$$V_n[\lambda].$$

The usual polynomial multiplication defines $\underline{a}(\lambda)$ as a linear map

$$V_n[\lambda] \to V_{n+1}[\lambda].$$

Explicitly,

$$\begin{aligned} &\underline{a}(\lambda)(v_0 + \lambda v_1 + \ldots + \lambda^n v_n) \\ &\quad = \alpha_0(v_0) + \lambda\alpha_0(v_1) + \ldots + \lambda^n\alpha_0(v_n) \qquad (7.2) \\ &\quad + \lambda\alpha_1(v_0) + \lambda^2\alpha_1(v_1) + \ldots + \lambda^{n+1}\alpha_1(v_n). \end{aligned}$$

<u>Theorem 7.1</u>. Suppose $(V, W, \underline{a})$ is a singular Kronecker system. Then, there is a non-zero $\underline{v} \in V[\lambda]$ such that:

$$\underline{a}(\underline{v}) = 0. \tag{7.3}$$

This is proved (as Theorem 3.2, of Chapter 3) in Volume III.

Definition. The smallest integer n such that there is a non-zero

$$\underset{\sim}{v} \in V_n[\lambda]$$

with $\underset{\sim}{g}(\underset{\sim}{v}) = 0$ is called the minimal Kronecker index of the system

$$(V, W, \underset{\sim}{g}).$$

Theorem 7.2. Suppose that n is the minimal Kronecker index of the Kronecker system $(V, W, \underset{\sim}{g})$, and $\underset{\sim}{v}(\lambda) = v_0 + \lambda v_1 + \ldots + \lambda^n v_n$ is an element of $V_n[\lambda]$ such that

$$\underset{\sim}{g}(\underset{\sim}{v}) = 0.$$

Then, the elements $v_0, \ldots, v_n$ of V are linearly independent. Further, let V' be the linear subspace of V spanned by $v_0, \ldots, v_n$. Then,

$$\alpha_0(V_1') = \alpha_1(V_0'). \tag{7.4}$$

Proof. The linear independence is proved in Theorem 4.1 of Chapter 3 of Volume III. (Notice that, if $n > 0$, its "minimality" implies there is no non-zero $v_0 \in V$ such that $\alpha_0(v_0) = \alpha_1(v_0) = 0$, so that the hypotheses of Theorem 4.1 are indeed satisfied.)

To prove (7.4), set the right hand side of (7.2) equal to zero, and equate to zero the coefficients of powers of λ.

This gives the following relations:

$$\begin{aligned} \alpha_0(v_0) &= 0 \\ \alpha_0(v_1) &= -\alpha_1(v_0) \\ \alpha_0(v_2) &= -\alpha_1(v_1) \\ &\vdots \\ \alpha_0(v_n) &= \alpha_1(v_{n-1}) \\ \alpha_0(v_n) &= 0. \end{aligned} \tag{7.5}$$

These relations obviously prove (7.4).

Theorem 7.3. Let n be the minimal Kronecker index of $(V, W, \mathfrak{g})$, and let V' be the linear subspace of V spanned by $v_0, \ldots, v_n$, as defined in the statement of Theorem 7.2. Let

$$W' = \alpha_0(V') = \alpha_1(V').$$

Then, (V', W') defines a subsystem of $(V, W, \mathfrak{g})$. The minimal Kronecker index of the quotient system

$$(V, W, \mathfrak{g})/(V', W', \mathfrak{g}')$$

is no less than n.

Proof. Set:

$$V'' = V/V', \; W'' = W/W'.$$

Let

$$\underset{\sim}{g}''(\lambda)\colon V'' \to W''$$

be the quotient map of $\underset{\sim}{g}(\lambda)\colon V \to W$.

Suppose the minimal Kronecker index of the quotient system is less than n. There is then a non-zero

$$\underset{\sim}{v}'' \in V_{n-1}''[\lambda],$$

such that

$$\underset{\sim}{g}''(\underset{\sim}{v}'') = 0. \tag{7.6}$$

Let $\underset{\sim}{v} \in V_{n-1}[\lambda]$ be a polynomial which goes into v' under the quotient map

$$V \to V'' \equiv V/V'.$$

Then, (7.6) means that:

$$\underset{\sim}{g}(\underset{\sim}{v}) = \underset{\sim}{W}' \in W_n'[\lambda]. \tag{7.7}$$

<u>To prove</u>: <u>There is a</u> $\underset{\sim}{v}' \in V'_{n-1}[\lambda]$ <u>such that</u>:

$$\underset{\sim}{g}(\underset{\sim}{v}') = \underset{\sim}{W}'. \tag{7.8}$$

First, let us note that proving (7.8) will finish the proof of (7.8). For, combining (7.7) and (7.8) gives:

$$\underset{\sim}{g}(\underset{\sim}{v} - v') = 0, \tag{7.9}$$

while the non-vanishing of $\underset{\sim}{v}''$ implies that $\underset{\sim}{v} - \underset{\sim}{v}' \neq 0$. But, $\underset{\sim}{v} - \underset{\sim}{v}'$ is a <u>polynomial of degree n-1</u>, and hence relation (7.9) violates the minimality of n.

Turn to the proof of (7.8). Note that:

$$\dim V' = n + 1$$

$$\dim W' \leq n.$$

Hence,

$$\begin{aligned} &\dim V'_{n-1}[\lambda] = n(n + 1) \\ &\dim W_n'[\lambda] \leq n(n + 1) = \dim V'_{n-1}. \end{aligned} \tag{7.10}$$

Consider g as defining a linear map between the following vector spaces

$$g: V'_{n-1}[\lambda] \to W_n'[\lambda]. \tag{7.11}$$

The map g <u>must be one-one</u>. Otherwise, there would be a polynomial of degree (n-1) which is annihilated by g, contradicting the minimality of n.) Condition (7.10) then implies that the map (7.11) is <u>onto</u>, which is precisely what is asserted by (7.8).

<u>Remark</u>. This argument proves more, namely that:

$$n = \dim W' = \dim V' - 1, \tag{7.12}$$

which was not a-priori obviously. In particular, we see that:

$$\alpha_0(v_1), \ldots, \alpha_0(n)$$

are <u>linearly independent elements of W</u>.

Definition. The system $(V', W', \underset{\sim}{g}')$ will be called a minimal Kronecker subsystem of $(V, W, \underset{\sim}{g})$. (It is not unique, of course.)

Here is the main result of the Kronecker Theory.

Theorem 7.4. Let n be the minimal Kronecker index of the Kronecker system $(V, W, \underset{\sim}{g})$, and let $(V', W', \underset{\sim}{g}')$ be a minimal Kronecker subsystem. Then, there is another subsystem $(V'', W'', \underset{\sim}{g}'')$ such that:

$$(V, W, \underset{\sim}{g}) = (V', W', \underset{\sim}{g}') \oplus (V'', W'', \underset{\sim}{g}''). \tag{7.13}$$

The minimal Kronecker index of $(V'', W'', \underset{\sim}{g}'')$ is not less than n.

Proof. We shall apply the general theory developed in Section 5. Start off with

$(V'', W'', \underset{\sim}{g}'')$ as the quotient system

$(V, W, \underset{\sim}{g})/(V', W', \underset{\sim}{g}')$.

We know from Section 5 that the system splits as a direct sum (7.13) if there are linear maps

$A: V'' \to V'$

$B: W'' \to W'$

such that:

$$B\underset{\sim}{g}'' - \underset{\sim}{g}'A = \underset{\sim}{\ell} \tag{7.14}$$

where $\boldsymbol{\alpha}(\lambda)$ is, for each $\lambda \in C$, a linear map

$$V'' \to W''. \tag{7.15}$$

Also, $\beta(\lambda)$ is a linear polynomial, i.e. it is of the following form:

$$\beta(\lambda) = \beta_0 + \lambda\beta, \tag{7.16}$$

where β_0, β_1 are linear maps: $V'' \to W'$. Suppose:

$$\boldsymbol{\alpha}'' = \alpha_0'' + \lambda\alpha_1''$$

$$\boldsymbol{\alpha}' = \alpha_1' + \lambda\alpha_1'.$$

Then, (7.14) takes the following form:

$$\begin{aligned} B\alpha_0'' - \alpha_0'A &= \beta_0 \\ B\alpha_1'' - \alpha_1'A &= \beta_1. \end{aligned} \tag{7.17}$$

Consider the map

$$L(V'', V') \oplus L(W'', W') \to L(V'', W') \oplus L(V'', W') \tag{7.18}$$

defined by formula (7.17), i.e.

$$(A, B) \to (B\alpha_0'' - \alpha_0'A,\ B\alpha_1'' - \alpha_1'A). \tag{7.19}$$

To prove the existence of (A, B), hence to finish the proof of Theorem 7.4, one must prove that the map (7.18) is <u>onto</u>. We shall do this by estimating the dimensions of the spaces involved. Set:

$$\dim V' = n + 1 \tag{7.20}$$

$$\dim V'' = m_1 \tag{7.21}$$

$$\dim W'' = m_2. \tag{7.22}$$

It is known that:

$$\dim W' = n \tag{7.23}$$

$$k = \text{dimension of kernal of the map (7.19).} \tag{7.24}$$

Then, the dimension of the domain of the map (7.18) is:

$$m_1(n+1) + m_2 n.$$

The dimension of the space on the right hand side of (7.18) is:

$$2m_1 n.$$

Hence, the map (7.19) is onto (hence Theorem 7.4 would be proved) if and only if the following condition is satisfied:

$$m_1(n+1) + m_2 n - k = 2m_1 n,$$

or

$$\begin{aligned} k &= m_2 n + m_1 - m_1 n \\ &= (m_2 - m_1)\, n + m_1. \end{aligned} \tag{7.25}$$

On the other hand, k is the dimension of the subspace if

$$L(V'', V') \oplus L(W'', W')$$

consisting of the pairs (A, B) of linear maps such that:

$$\begin{aligned} B\alpha_0'' - \alpha_0' A &= 0 \\ B\alpha_1'' - \alpha_1' A &= 0 \end{aligned} \tag{7.26}$$

The proof now proceeds by showing that the solutions (A, B) of equations (7.26) have the dimension given by formula (7.25). This is in fact done by Gantmacher in Section 3 of Chapter XII of [1], by converting (7.26) into matrix equations and using Theorem 7.3 to show that the resulting equations have the correct rank. Since I have not not yet succeeded in giving a more understandable proof of this (as I hope to do later) I will defer, at least for the moment, to Gantmacher's proof.

8. DECOMPOSITION OF KRONECKER SYSTEMS INTO DIRECT SUM OF CYCLIC AND DUAL CYCLIC SUBSYSTEMS

We now apply the basic results proved in previous sections to decomposition of Kronecker systems. First, we recapitulate notations and provide some more basic definitions.

Let V, W be complex finite dimensional vector spaces, and

$$\lambda \to \underset{\sim}{\alpha}(\lambda) = \alpha_0 + \lambda\alpha_1 \in L(V, W)$$

be a linear polynomial map which defines

$$(V, W, \underset{\sim}{\alpha})$$

as a Kronecker system. The system is said to be regular if there is a $\lambda_0 \in C$ such that

$$\underset{\sim}{g}(\lambda_0) \text{ is an isomorphism between } V \text{ and } W.$$

If a system is not regular, it is said to be singular.

If $\underset{\sim}{g}(\lambda)$ is not one-one for all $\lambda \in C$, there is an integer n and a non-zero polynomial

$$\underset{\sim}{v} \in V_n[\lambda]$$

such that

$$\underset{\sim}{g}\underset{\sim}{v} = 0.$$

The minimal such integer n is called the minimal Kronecker index of the system.

Definition. The singular system (V, W, $\underset{\sim}{g}$) is said to be cyclic if its minimal Kronecker index is n, and:

$$\dim V = n+1 \tag{8.1}$$

$$\dim W = n. \tag{8.2}$$

Remark: It follows from the results of Section 7 that "cyclicity" means that there is a basis

$$(v_0, \ldots, v_n)$$

of V such that:

$$\mathfrak{g}(v_0 + \lambda v_1 + \ldots + \lambda^n v_n) = 0 \tag{8.3}$$

$$\alpha_0(v_1), \ldots, \alpha_0(v_n) \text{ is} \tag{8.4}$$

a basis for W.

<u>Theorem 8.1</u>. If a Kronecker system is cyclic, it is also irreducible in the sense that it cannot be written as the direct sum of subsystems.

<u>Proof</u>. Suppose otherwise, i.e. (8.1) and (8.2) are satisfied,

$$V = V' \oplus V''$$

$$W = W' \oplus W''$$

$$\mathfrak{g}(V') \subset W'$$

$$\mathfrak{g}(W'') \subset W''$$

Now, because of (8.2), i.e. the fact that $\mathfrak{g}(\lambda)$ <u>reduces dimension</u>, $\mathfrak{g}$ restricted to V' <u>or</u> V" is singular. This means that there would be a polynomial on V' or V" of degree less than n which would be annihilated by $\mathfrak{g}$, i.e. the minimal Kronecker index of (V, W, $\mathfrak{g}$) would be less than n, contradiction, and Theorem 8.1 is proved.

Now we can proceed to decompose a general singular system (V, W, $\mathfrak{g}$) into direct sum of cyclic ones, with the aid of Theorem 7.4. Suppose first that:

$\underset{\sim}{g}(\lambda)$ is not one-one for all $\lambda \in C$.

Then, applying Theorem 7.4,

$$(V, W, \underset{\sim}{g}) = (V', W', \underset{\sim}{g}') \oplus (V'', W'', \underset{\sim}{g}''),$$

where:

$(V', W', \underset{\sim}{g}')$ is cyclic, of minimal Kronecker index n'.

We see that <u>either</u>:

$(V'', W'', \underset{\sim}{g}'')$ has a minimal Kronecker index, say $n'' \geq n''$,

<u>or</u> $\underset{\sim}{g}''(\lambda_0)$ is one-one for at least one $\lambda_0 \in C$.

If the former case holds, apply the decomposition Theorem 7.4 to $(V'', W'', \underset{\sim}{g}'')$. Continuing in this way, we have:

<u>Theorem 8.2</u>. A Kronecker system $(V, W, \underset{\sim}{g})$ can be written as the direct sum of cyclic Kronecker systems, and a Kronecker system such that

$\underset{\sim}{g}(\lambda_0)$ is one-one for at least one $\lambda_0 \in C$. (8.5)

Let us now study Kronecker systems satisfying (8.5). To this end, it is convenient to introduce the following notion.

Definition. Let $(V, W, \mathfrak{g})$ be a system. The dual system $(V', W', \mathfrak{g}')$ is that defined by the following formulas:

$$\begin{aligned} V' &= W^d \equiv \text{the dual space to } W \\ W' &= V^d \equiv \text{the dual space to } V. \\ \mathfrak{g}'(\lambda) &= \mathfrak{g}(\lambda)^d \equiv \text{the dual map to} \\ &\qquad \mathfrak{g}(\lambda)\colon V \to W. \end{aligned} \tag{8.6}$$

We denote this dual system as follows:

$$(V, W, \mathfrak{g})^d \equiv (W^d, V^d, \mathfrak{g}^d). \tag{8.7}$$

Definition. A Kronecker system is cocyclic if it is the dual of a cyclic Kronecker system.

Suppose now that $(V, W, \mathfrak{g})$ is a Kronecker system which satisfies (8.5). Consider its dual system $(V, W, \mathfrak{g})^d$. From linear algebra we know that:

$\mathfrak{g}(\lambda)^d$ is one-one if and only if $\mathfrak{g}(\lambda)$ is onto.

Hence, we have:

Theorem 8.3. If $(V, W, \mathfrak{g})$ satisfies (8.5), then $(V, W, \mathfrak{g})^d$ also satisfies (8.5) if and only if the system $(V, W, \mathfrak{g})$ is regular.

Suppose then that $(V, W, \mathfrak{g})$ is not regular, but does satisfy (8.5). By Theorem 8.3, its dual system $(V, W, g)^d$ does not satisfy (8.5), i.e.

$$g(\lambda)^d: W^d \to V^d$$

is not one-one for all $\lambda \in C$. Hence, Theorem 8.2 can be applied again. We can write:

$$(V, W, g)^d = (V', W', g') \oplus (V'', W'', g''), \qquad (8.8)$$

where:

(V', W', g') is the direct sum of cyclic systems

$g''(\lambda): V'' \to W''$ is one-one for at least one $\lambda_0 \in C$. (8.9)

Let:

$(V')^{\perp}$ = orthogonal complement of $V' \subset W^d$ in W $\equiv \{W \in W: \theta(w) = 0 \text{ for all } \theta \in W'\}$.

Then, from linear algebra, we know that,

$$W = (V')^{\perp} \oplus (V'')^{\perp}$$

$$V = (W')^{\perp} \oplus (W'')^{\perp}$$

$$((W')^{\perp}) \subset (V')^{\perp}$$

$$((W'')^{\perp}) \subset (V')^{\perp}.$$

We are assuming that (8.5) is satisfied, hence:

$$\mathfrak{g}(\lambda): (W')^{\perp} \to (V')^{\perp}$$

and

$$\mathfrak{g}(\lambda): (W'')^{\perp} \to (V'')^{\perp}$$

are both one-one. Using (8.9), we see that

$$\mathfrak{g}(\lambda_0): (W')^{\perp} \to (V')^{\perp}$$

is an isomorphism.

Putting all these decompositions together, we have:

Theorem 8.4. (Kronecker Structure Theorem). Let $(V, W, \mathfrak{g})$ be a Kronecker system. Then, it can be written as a direct sum of cyclic,cocyclic, and regular systems.

Remark and exercise. The argument given above should be regarded as only a sketch of a proof. Give a precise proof, for example by induction on the dimension of V.

Remark: Notice the analogy with the so-called "fundamental theorem of finitely generated abelian groups," e.g. in VOLUME I, which asserts that every finitely generated abelian group is the direct sum of finite and infinite cyclic groups. The "regular" Kronecker systems are analogous to the "infinite cyclic groups," while the cyclic and cocyclic systems are analogous to the "finite cyclic" groups.

"Structure theorems" like this are always in close algebraic relation to "isomorphism" theorems. We shall now briefly discuss such notions for Kronecker systems.

9. VECTOR SPACE AND MODULE ISOMORPHISM OF KRONECKER SYSTEMS

First, let us give the relevant general definitions for arbitrary systems.

Definition. Let $(V, W, \mathfrak{g})$, $(V', W', \mathfrak{g}')$ be two systems. A pair (A, B)

$$A: V \to V'$$

$$B: W \to W'$$

of linear maps is said to define an isomorphism of vector space type between the two systems of the following conditions are satisfied:

A, B are isomorphisms between these vector spaces on which they are defined. (9.1)

$$B\mathfrak{g}(\lambda) = \mathfrak{g}'(\lambda)A \quad \text{for all } \lambda \in C, \tag{9.2}$$

i.e. the following diagram of maps is commutative:

$$\begin{tikzcd} V \arrow[r, "\mathfrak{a}"] \arrow[d, "A"'] & W \arrow[d] \\ V' \arrow[r, "\mathfrak{a}'"] & W' \end{tikzcd}$$

<u>Remark</u>: This is the natural notion of "isomorphism" from the point of view of differential equations. Consider the system of ordinary, linear, constant coefficient differential equations associated with the two systems.

$$\mathfrak{a}(\tfrac{d}{dt})(v(t)) = 0 \tag{9.3}$$

$$\mathfrak{a}'(\tfrac{d}{dt})(v'(t)) = 0. \tag{9.4}$$

Suppose that (9.2) is satisfied. Given a curve $t \to v(t)$ in V which is a solution of differential equation (9.3), set:

$$v'(t) = A(v(t)). \tag{9.5}$$

Suppose that:

$$\mathfrak{a} = \alpha_0 + \lambda\alpha_1 + \cdots$$

$$\mathfrak{a}' = \alpha_0' + \lambda\alpha_1 + \cdots$$

(9.2) requires that:

$$B\alpha_0 = \alpha_0'A,\ B\alpha_1 = \alpha_1'A,\ \ldots \tag{9.6}$$

Hence,

$$\underset{\sim}{\alpha}'(\tfrac{d}{dt})(v') = \alpha_0'(v') + \alpha_0' \frac{d}{dt} v' + \ldots$$

$$= \alpha_0' A v + \alpha_0' A \frac{dv}{dt} + \ldots$$

$$=, \text{ using (9.6)},$$

$$B\alpha_0 v + B\alpha_0 \frac{dv}{dt} + \ldots$$

$$= B\underset{\sim}{\alpha}(\tfrac{d}{dt})(v) = 0.$$

In particular, we see that the correspondence

$$v(t) \to Av(t)$$

maps solutions of (9.3) into solutions of (9.4). Since A is a vector space isomorphism, this is an "isomorphism" between the solutions of the two differential equations.

Let us turn now to consider an isomorphism notion which is natural from the "module" point of view. Let $V[\lambda]$ denotes the polynomials with coefficients in V. If $(V, W, \underset{\sim}{\alpha})$ is a system, , acting in the usual way, maps $\underset{\sim}{\alpha}$: $V[\lambda] \to W[\lambda]$. For example, if:

$$\underset{\sim}{\alpha} = \alpha_0 + \lambda\alpha_1 + \ldots,$$

$$\underset{\sim}{v} = v_0 + \lambda v_1 + \ldots,$$

then

$$\underset{\sim}{\alpha}(\underset{\sim}{v}) = \alpha_0(v_0) + (\alpha_1(v_0) + \alpha_0(v_1))\lambda + \ldots$$

Let $C[\lambda]$ denote the ring of polynomials with complex numbers as coefficients. The natural multiplication makes

$V[\lambda]$ and $W[\lambda]$ into $C[\lambda]$-modules.

<u>Definition</u>. The systems $(V, W, \underset{\sim}{g})$, $(V', W', \underset{\sim}{g}')$ are said to be <u>isomorphic in the module sense</u> if there are $C[\lambda]$-module isomorphisms

$$\underset{\sim}{A}: V[\lambda] \to V'[\lambda]$$

$$\underset{\sim}{B}: W[\lambda] \to W'[\lambda]$$

such that:

$$\underset{\sim}{B}\underset{\sim}{g} = \underset{\sim}{g}'\underset{\sim}{A}. \tag{9.7}$$

<u>Remark</u>: Instead of our terminology, Gantmacher [1] uses the terms "strong equivalence" and "equivalence." Similiar notions appear in Rosenbrock's seminal treatise on systems theory [1].

Notice that no longer is the transformation $\underset{\sim}{A}$ related (at least in an obvious way) to an isomorphism between the solutions of differential equations (9.3) and (9.4). Of course, there are relations at the "Laplace transform" level.

Here is another useful way of putting the conditions. A module homomorphism

$$\underset{\sim}{A}: V[\lambda] \to V'[\lambda]$$

can also be defined via an element

$$\underset{\sim}{A} \in L(V, V')[\lambda].$$

Thus, condition (9.7) becomes:

$$\underset{\sim}{B}(\lambda)\underset{\sim}{g}(\lambda) = \underset{\sim}{g}'(\lambda)\underset{\sim}{A}(\lambda) \tag{9.8}$$

for all $\lambda \in C$.

Here is a basic result linking the two isomorphism notions, in the "Kronecker" case:

<u>Theorem 9.1</u>. Suppose $(V, W, \underset{\sim}{g})$, $(V, W, \underset{\sim}{g}')$ are <u>regular</u> Kronecker systems, and

$$\underset{\sim}{A} \in L(V, V)[\lambda]$$

$$\underset{\sim}{B} \in L(W, W)[\lambda]$$

satisfy the following conditions:

$$\underset{\sim}{B}(\lambda)\underset{\sim}{g}(\lambda) = \underset{\sim}{g}'(\lambda)\underset{\sim}{A}(\lambda) \tag{9.9}$$

for all $\lambda \in C$.

$$\det(\underset{\sim}{A}(\lambda)) \equiv \text{non-zero constant} \tag{9.10}$$

$$\det(\underset{\sim}{B}(\lambda)) \equiv \text{non zero constant.} \tag{9.11}$$

Then, $(V, W, \underset{\sim}{g})$, $(V, W, \underset{\sim}{g}')$ are vector space isomorphic.

The proof is given in Theorem 6 of Chapter VI of Gantmacher [1]. Gantmacher also discusses the relation between the Weirstrassian elementary division theory for $\underset{\sim}{g}$, and the various isomorphism notions.

<u>Exercise</u>. Translate these results described by Gantmacher

in matrix language into the more algebraic language used here.

Exercise. Show that regular Kronecker systems can be decomposed into the direct sum of regular systems with one-dimensional V and W. (This should be done both directly, and using Theorem 9.1. In fact, it is a corollary of Theorem 9.1 and the Weirstrassian elementary divisor theory.

10. SOLVING INHOMOGENEOUS LINEAR, CONSTANT COEFFICIENT ORDINARY DIFFERENTIAL EQUATIONS BY MEANS OF THE KRONECKER STRUCTURE THEORY

Let (V, W, $\mathfrak{g}$) be a Kronecker system. Suppose we want to solve

$$\mathfrak{g}\left(\frac{d}{dt}\right)(v(t)) = w(t), \qquad (10.1)$$

given the curve $t \to w(t)$ in W. The Kronecker structure theory can be obviously used here.

The system (V, W, $\mathfrak{g}$) can be written as the direct sum

$$(V^1, W^1, \mathfrak{g}^1) \oplus \ldots \oplus (V^m, W^m, \mathfrak{g}^m)$$

of subsystem, each of which is cyclic, cocyclic or regular. Thus, w(t) and v(t) can be decomposed as a direct sum:

$$w(t) = w_1(t) \oplus \ldots \oplus w_m(t)$$

$$v(t) = v_1(t) \oplus \ldots \oplus v_m(t),$$

and (2.1) can be written as:

$$g(\frac{d}{dt})(v_i) = w_i \tag{10.2}$$

for $1 \le i \le m$.

Thus, the solution of (10.1) depends on the solution of (10.1) in the three cases, cyclic, cocyclic, and regular. Let us examine these in turn.

Case 1. (V, W, g) is cyclic.

Set:

$$g = \alpha_0 + \lambda\alpha_1.$$

In this case, we know that:

$$\alpha_0(V) = \alpha_1(V) = W. \tag{10.3}$$

Hence (see Theorem 2.1 of Chapter 3 of Volume 3, or prove it directly) there is a

$$\gamma \in L(V, V)$$

such that:

$$\alpha_0 = \alpha_1\gamma. \tag{10.4}$$

Hence, (10.1) takes the following form:

$$\alpha_1(\gamma v + \frac{dv}{dt}) = w(t). \tag{10.4}$$

Because of (10.3), we see that differential equation (10.4) is - hence also (10.1) - is solvable for $v(t)$.

<u>Exercise</u>. In terms of a Kronecker basis $(v_0, \ldots, v_n)$ for V, write down the explicit solution of (10.4).

<u>Case 2</u>. $(V, W, \underset{\sim}{g})$ is cocyclic

By definition, the dual system

$$(V, W, \underset{\sim}{g})^d \equiv (W^d, V^d, \underset{\sim}{g}^d)$$

is cyclic. Hence, there is a map

$$\gamma^d \in L(W^d, W^d)$$

such that:

$$\alpha_0{}^d \gamma^d = \alpha_1{}^d. \tag{10.5}$$

Hence,

$$\alpha_1 = \gamma\alpha_0, \tag{10.6}$$

where γ is a linear map: $W \to W$. Hence, in this case, (10.1) takes the following form:

$$\gamma\alpha_0 \frac{dv}{dt} + \alpha_0 v = w. \tag{10.7}$$

We shall now have a differential equation which the curve $t \to w(t)$ on the right hand side of (10.7) must satisfy. This will give the "integrability conditions" for the system (10.1), in this cocyclic case.

Suppose:

$$\dim V = n.$$

Then, since (W^d, V^d, g^d) is cyclic,

$$\dim W^d = n+1,$$

hence also:

$$\dim W = n+1.$$

We know also that:

$$(\gamma^d)^{n+1} = 0.$$

hence:

$$\gamma^{n+1} = 0. \tag{10.9}$$

Let D be the following differential operator:

$$D = -\gamma \frac{d}{dt}. \tag{10.10}$$

From (10.9), we have:

$$D^{n+1} = 0 \tag{10.11}$$

(10.7) can be rewritten as follows

$$(D + 1)(\alpha_0 v) = W. \tag{10.12}$$

Now, using (10.11)

$$\frac{1}{I-D} = 1 + D + D^2 + \ldots + D^n + 0. \tag{10.13}$$

<u>Define</u> $\frac{1}{I-D}$

as the n-th order differential operator on the right hand side of (10.13). Then, (10.12) implies that:

$$\alpha_0 v = (1-D)^{-1}(w). \tag{10.14}$$

Let

$$\beta: W \to W/\alpha_0(V) \tag{10.15}$$

be the quotient vector space map. Then, (10.14) implies that:

$$\begin{aligned} 0 &= \beta(1-D)^{-1}(w) \\ &= \beta(w) + \beta D(w) + \ldots + \beta D^n w \end{aligned} \tag{10.16}$$

This is the differential equation that we need. Here is the precise result:

Theorem 10.1. Suppose $(V, W, \underset{\sim}{g})$ is a cocyclic system, whose dual system has Kronecker index n. Then, the inhomogeneous differential equation (10.1) has a solution if and only if the given inhomogeneous term $t \to w(t)$ satisfies the differential equation (10.16). In other words, the differential equations (10.16) are the "integrability conditions" for the differential equation (10.1).

Proof. We have already seen that (10.16) are necessary conditions for the solvability of (10.1). Conversely, suppose that $t \to w(t)$ solves (10.14).

Now, (10.16) means that there is a curve $t \to v(t)$ in V such that:

$$\alpha_0(v) = (1-D)^{-1}(w). \tag{10.17}$$

Apply the differential operator (1-D) to both sides of (10.17), showing that v is a solution of (10.1). Notice that v is obtained from w by differentiations and solutions of linear algebraic equations.

The remaining case is easy.

Exercise. Suppose that the Kronecker system (V, W, $\underset{\sim}{g}$) is regular. Show that the differential equation (10.1) can be solved, given $t \to w(t)$, and find the explicit formulas.

Remark: Our conclusions in this section may be readily stated in terms of the general formalism of the theory of linear differential operators. (See GPS, Chapter I). Let Γ, Γ' be the cross-section spaces of two vector bundles over a manifold M. (In the specific case treated in this section, M = R, the space of the variable t). Let

$$D: \Gamma \to \Gamma$$

be a linear differential operator. A set of integrability conditions for the system (Γ, Γ', D) consists of a space Γ'' of cross-sections of another vector bundle, and a differential operator:

$$D': \Gamma' \to \Gamma''$$

such that:

$D'D = 0$, i.e.

image D = kernel D'

In a sufficiently small region of the base manifold,

image D = kernel D'.

Typically, the "integrability conditions" can be iterated, i.e. a differential operator

$$D'': \Gamma'' \to \Gamma''$$

is obtained which are integrability conditions for D. The result is typically a sequence of differential operators:

$$\Gamma \xrightarrow{D} \Gamma' \xrightarrow{D'} \Gamma'' \xrightarrow{D''} \Gamma''' \to \dots,$$

called a <u>resolution</u> of the operator d. For example, for the case where:

M is any manifold,

$\Gamma = F(M) = C^\infty$ functions

$\Gamma' = F'(M)$ = 1-differential operators,

$D = d$ = exterior derivative

the Γ'', Γ''', are the higher degree differential forms, and the operators D', D'',... are the exterior derivative operators. D. C. Spencer and his many co-workers have developed an extensive theory of such resolutions. See the recent book by Kumpera and Spencer, "Lie Equations," Princeton University Press.

Exercise. In case of the differential operator (10.1), sketch how such a "resolution" may be constructed, using the Kronecker theory.

11. INPUT-OUTPUT SYSTEMS AND THE KRONECKER THEORY

In previous sections, we have described the beautiful Kronecker algebraic "structure" theory for systems of differential equations of the form:

$$\alpha_1 \frac{dv}{dt} + \alpha_0 v = 0, \tag{11.1}$$

where α_0, α_1 are linear maps

$$V \to W$$

between vector spaces. In Volume VIII I have described an algebraic theory for input-output systems of ordinary differential equations with constant coefficients.

Now, the input-output systems are special cases of the Kronecker systems (11.1). Presumably, the Kronecker theory, when specialized to systems of input-output type, should give results which are compatible with the results described in Volume VIII. The aim of this section is to write down this specialization. A deeper analysis must wait for a later time. I believe that the system-theory results of Rosenbrock [1] may be elegantly fitted into this framework.

First, we shall describe what is meant by an "input-output system." Let X, U, Y be complex, finite dimensional real vector spaces, called the _state_, _input_, and _output spaces_. Recall that an _input-output system_ (of differential equations) is one of the form:

$$\begin{aligned} \frac{dx}{dt} &= f(x, u, t) \\ y &= g(x, u, t), \end{aligned} \tag{11.2}$$

to be solved for a curve

$$t \to (x(t), u(t), y(t))$$

in $X \times U \times Y$. f,g are mappings (in general, non-linear) of

$$X \times U \times T \to Y.$$

We shall specialize so that f,g are of the following form:

$$\begin{aligned} f(x, u, t) &= Ax + Bu \\ g(x, u, t) &= Cx + Du, \end{aligned}$$

where A, B, C, D are maps where domain and ranges are indicated by the following (non-commutative) diagram of maps:

$$\begin{tikzcd} U \arrow[r, "D"] \arrow[d, "B"'] & Y \\ X \arrow[r, "A"] & X \arrow[u, "C"'] \end{tikzcd}$$

To convert the system (11.2) into one of Kronecker type, (11.1), proceed as follows. Set:

$$V = X \oplus U \oplus Y \tag{11.3}$$

$$W = X \oplus Y. \tag{11.4}$$

Define α_0, α_1 as linear maps

$$V \to W$$

as follows:

$$\alpha_0(x \oplus u \oplus y) = (Ax + Bu) \oplus (Cx + Du - y) \tag{11.5}$$

$$\alpha_1(x \oplus u \oplus y) = -x. \tag{11.6}$$

Define $\underset{\sim}{\alpha} \in L(V, W)[\lambda]$ as follows:

$$\underset{\sim}{\alpha}(\lambda) = \alpha_0 + \lambda\alpha_1. \tag{11.7}$$

Then, for a curve

$$t \to v(t) \equiv x(t) \oplus u(t) \oplus y(t)$$

in V,

$$\begin{aligned} \alpha(\tfrac{d}{dt})(v) &= (Ax + Bu) \oplus (Cx + Du - y) \\ &= \frac{dx}{dt}. \end{aligned} \tag{11.8}$$

Hence,

$$\alpha(\frac{d}{dt})(v) = 0 \text{ if and only if:}$$

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{11.9}$$

i.e. the curve $t \to (x(t), u(t), y(t))$ satisfies the equations (11.9) of the input-output system.

We recognize that (11.6)-(11.8) are the equations of a Kronecker system, as defined in the previous section. Let us proceed to specialize the ideas developed there (and by Gantmacher [1]) to this situation.

<u>Kronecker subsets of V</u>

We know that:

$$\dim V = \dim X + \dim Y + \dim U$$

$$\dim W = \dim X + \dim Y,$$

hence:

$$\dim V > \dim W.$$

In particular, since

$$\underset{\sim}{g}(\lambda) \text{ maps } V \to W,$$

$$\text{kernel } \underset{\sim}{g}(\lambda) \neq (0)$$

$$\text{for all } \lambda \in C.$$

Hence the system $\underset{\sim}{g}(\lambda)$ is a <u>singular system</u>, in the sense of Kronecker.

In particular (see Volume III, Section 4, Chapter 3) there are polynomials with coefficients in V,

$$\underset{\sim}{v}(\lambda) \in V[\lambda] \equiv v_0 + \lambda v_1 + \dots + \lambda^n v_n, \tag{11.10}$$

such that:

$$\underset{\sim}{g}(\lambda)(\underset{\sim}{v}(\lambda)) = 0 \tag{11.11}$$

for all λ.

The minimal integer n which occurs in this way is called the <u>first Kronecker index</u> of the system. It then follows (see Theorem 4.1, of Chapter of Volume III) that the elements $(v_0, \dots, v_n)$ of V are linearly independent. The form what is called a <u>Kronecker subset</u> of V.

Let us suppose now that $\underset{\sim}{g}$ is defined by (11.6)-(11.7). Set:

$$\underset{\sim}{v}(\lambda) = \underset{\sim}{x}(\lambda) \oplus \underset{\sim}{u}(\lambda) \oplus \underset{\sim}{y}(\lambda),$$

where:

$$\underset{\sim}{x} \in X[\lambda], \ \underset{\sim}{u} \in U[\lambda], \ \underset{\sim}{y} \in Y[\lambda].$$

<u>Remark</u>: Recall that $X[\lambda]$ denotes the polynomials in λ, with coefficients in the vector space X, or, alternately, the space of polynomial maps

$$C \to X.$$

Then,

$$\begin{aligned} g(y) &= \alpha_0(x \oplus y \oplus u) + \lambda\alpha_1(x \oplus y \oplus u) \\ &= (Ax + Bu) \oplus (Cx + Du - y) \qquad (11.12) \\ &- \lambda x. \end{aligned}$$

In particular, we see that:

$$g(y) = 0 \text{ if and only if}$$

$$(\lambda\text{-}A)x + Bu \qquad (11.13)$$

$$y = Cx + Du, \qquad (11.14)$$

(11.13) and (11.14) <u>together</u> imply that

$$y = C(\lambda\text{-}A)^{-1} Bu + Du. \qquad (11.15)$$

We recognize that this is the "Laplace transform" form of the relation between input and output for the system (11.9). In particular

$$\lambda \to C(\lambda\text{-}A)^{-1}B$$

is the <u>frequency response function</u> of the system (11.9), which plays such a key role in the theory (as explained, for example, in Volumes III and VIII.) It is good to see that it appears in this natural way when the Kronecker theory is applied to the input-output system. We can now sum up as follows:

Theorem 11.1. Consider an input-output system of form (11.9). Write it in Kronecker form, (11.8). Then, the minimal Kronecker index of the system is the smallest integer n such that the following condition is satisfied:

There are polynomials

$\underset{\sim}{x} \in X[\lambda]$

$\underset{\sim}{u} \in U[\lambda]$

$\underset{\sim}{y} \in Y[\lambda]$

of degree n satisfying equations (11.13) and (11.14).

Let us analyze conditions (11.13)-(11.14) further. Suppose that:

$$\dim X = m. \tag{11.16}$$

$$(A-\lambda)^{-1} = \frac{N(\lambda)}{d(\lambda)}, \tag{11.17}$$

where:

$$d(\lambda) \in C_m[\lambda],$$

i.e. $d(\lambda)$ is a scalar polynomial of degree m (e.g. the characteristic polynomial of the linear map $A: X \to X$), and

$$N(\lambda) \in L(X, X)_{m-1}[\lambda].$$

(In terms of matrices, $N(\lambda)$ may be taken as the "adjoint" matrix (i.e. that appearing in Cramer's rule) of $A-\lambda$).

Then,

$$\underset{\sim}{x} = \frac{N(\lambda)\underset{\sim}{u}(\lambda)}{d(\lambda)}$$

In particular, we see that we can always take $\underset{\sim}{u}$ to be a polynomial of degree m which is divisible by $d(\lambda)$, and satisfy equations (11.13)-(11.14). Hence, we have the following result.

Theorem 11.2. The minimal Kronecker index of the system (11.9) is no greater than the dimension of the state space X.

Chapter IX

THE GEOMETRY OF INTERACTIONS IN THERMODYNAMICS

1. INTRODUCTION

In my book "Geometry, Physics and Systems," Chapter 6, I have presented briefly a geometric setting for the ideas of classical equilibrium thermodynamics. Here, I will develop this approach further, and penetrate more deeply into the notion of "interaction" of thermodynamics systems. As we shall see, there are interesting mathematical connections with the theory of "interaction" of mechanical systems.

First, let us review some of the ideas presented in "Geometry, Physics and Systems."

2. SIMPLE SMOOTH THERMODYNAMICS SYSTEMS AND EQUATIONS OF STATE

Let M be the Euclidean space R^5. For purposes of identification with chemistry, label the coordinates of M by

$$U, T, S, P, V.$$

U is <u>internal energy</u>, T is <u>temperature</u>, S is <u>entropy</u>, P is <u>pressure</u>, V is <u>volume</u>. Let θ be the following one form on M, called the <u>Gibbs one-form</u>:

$$\theta = dU - TdS - PdV \tag{2.1}$$

A <u>simple smooth equilibrium thermodynamic system</u> is a C^∞ submanifold mapping

$$\varphi\colon R^2 \to M$$

such that:

$$\varphi^*(\theta) = 0.$$

Typically, such submanifolds are defined by setting certain functions on M equal to zero. These relations are called <u>equations of state</u>.

<u>Example</u>. <u>Ideal Gases</u>.

We follow Appendix D of Callen's textbook on Thermodynamics [1]. Start off with U, V as independent variables, (so that S, T, P are functions of them) and a relation of the following form:

$$S = f(U) + \log(C_1 V + C_2), \tag{2.2}$$

where C_1, C_2 are real constants, f( ) is an arbitrary function of one variable. Then,

$$dS = f'(U)dU + \frac{C_1 dV}{C_1 V + C_2}$$

$$= \frac{1}{T}\, dU + \frac{P}{T}\, dV. \tag{2.3}$$

Hence:

$$f'(U) = \frac{1}{T} \tag{2.4}$$

$$\frac{P}{T} = \frac{C_1}{C_1 V + C_2},$$

$$P(C_1 V + C_2) = C_1 T. \tag{2.5}$$

Equations (2.4) and (2.5) will be recognized as the equations of state of an "ideal gas," in the traditional form. (2.4) says that temperature is a function of internal energy. After appropriate normalization of the variables, (2.5) can be written as:

$$PV = RT, \tag{2.6}$$

which is the traditional form of the ideal gas law. (R is a constant).

Remark. Of course, this approach to thermodynamics is implicit (and, in fact, almost explicit, modulo a lack of suitable language available to him at the time) in Gibbs famous work on thermodynamics. It is an interesting historical question (to which I do not know the answer) whether

Gibbs knew of the mathematical material on Pfaffian equations, then in full development in France and Germany. (Gibbs in fact studied mathematics in France for a year or so. His later development of what is now called "vector analysis", which is based mathematically on Grassman's work, also indicates that he must have been aware of these trends in mathematics.) The best modern reference to this approach to thermodynamics is Weinreich's book [1], which is a definite improvement on the text by Callan on which I based my treatment in GPS.

We see then that a key mathematical ingredient in classical thermodynamics is the theory of Pfaffian equations. We shall now review this material.

3. MAXIMAL INTEGRAL SUBMANIFOLDS OF EXTERIOR SYSTEMS GENERATED BY ONE PFAFFIAN FORM

Turn to a general manifold M, and a differential ideal I of differential forms on M. An <u>integral manifold</u> of I is a submanifold map

$$\varphi: N \to M$$

such that:

$$\varphi^*(I) = 0.$$

Such a submanifold map is called a <u>maximal integral sub-</u>

manifold if, locally, it is contained in no integral submanifold of larger dimension.

The general ideas of the theory were developed gradually through the 19th century, and brought to completion in E. Cartan's work. One can find a more contemporary version in my review paper, "E. Cartan's geometric theory of partial differential equations," and in "Geometry, physics and systems."

Let I be generated by a single 1-form θ. Note that for historical reasons, a one-form is often called a Pfaffian form, and the differential equations defining the maximal integral submanifold are called Pfaffian equations. (In the 19th century literature, the associated "problem" is called the Problem of Pfaff.)

For each point $p \in M$, let

$$H_p = \{v \in M_p\colon \theta(v) = 0\}.$$

Let us suppose that the following non-singularity condition is satisfied:

$$\begin{aligned} &\dim (H_p) = \text{constant} \\ &\text{as } p \text{ ranges over } M. \end{aligned} \tag{3.1}$$

Set:

$$C_p = \{v \in H_p\colon v \lrcorner\, d\theta = 0\}. \tag{3.2}$$

<u>Definition</u>. The tangent vectors to M which lie in C_p are called the <u>Cauchy characteristic tangent vectors</u> of the ideal I.

As another non-singularity condition, suppose that:

$$\dim C_p = r = \text{constant as } p \text{ ranges over } M. \tag{3.3}$$

Then, the 2-form $d\theta$ passes to the quotient to define a <u>non-degenerate</u>, skew-symmetric form on the quotient vector space

$$H_p/C_p. \tag{3.4}$$

From linear algebra, we know that:

$$\dim (H_p/C_p) \text{ is even, say } 2n \tag{3.5}$$

$$\text{The maximal dimension of the linear subspaces of } H_p/C_p \text{ on which } d\theta \text{ is zero is } n. \tag{3.6}$$

Putting (3.3)-(3.6) together, we have:

$$\text{The maximal dimension of the linear subspaces of } H_p \text{ on which } d\theta \text{ is zero is } r + n. \tag{3.7}$$

<u>Exercise</u>. Prove (3.4)-(3.7).

<u>Remark</u>: We can interpret these algebraic results in terms of the "language" of the general theory of exterior systems. For each integer j, let $G^j(I)$ denote the space of integral elements of dimension j of the ideal I, i.e. the set of j-dimensional linear subspaces γ of the tangent bundle such that:

$$\omega(I) = 0$$

for all $\omega \in I$.

Then, (3.7) says that the maximal j for which $G^j(I)$ is non-empty is

$$j = r+n.$$

The general Cartan existence theorem for exterior systems then asserts (if the non-singularity conditions (3.1) and (3.3) are satisfied) that there are integral submanifolds of dimension (r + n), i.e. that the system is <u>in involution</u> in dimension (r + n). These integral submanifolds of dimension (r + n) are then the <u>maximal</u> ones, in the sense of the definition given above.

Now, the <u>general</u> Cartan existence theorem requires that the data is real analytic. <u>In this case</u>, it can be extended to the C^∞ situation, using the method of Cauchy characteristics. We can now sum up these remarks in the following result, which is the classical Theorem of Pfaff in modern language.

Theorem 3.1. Let M be a manifold of dimension m, and let θ be a 1-form on M which is non-zero at each point of M. Let I be the differential ideal of forms generated by θ. Suppose that r is the integer defined by (3.3), i.e. the dimension of the space of Cauchy characteristic vectors (assumed to be constant.) Then, the maximal integral manifolds of I are of dimension

$$\frac{m + r - 1}{2}. \tag{3.8}$$

Remarks. One can go further in description of the structure of maximal integral submanifolds. Let G be the group of contact transformations on M, i.e. the diffeomorphisms $g: M \to M$ such that:

$$g^*(\theta) = f\theta$$

for some $f \in F(M)$.

One can then prove that locally G acts transitively on the set of maximal integral submanifolds. I believe that this theorem should be credited to Sophus Lie, although I have not made a study on this point.

There is another way of calculating (3.8) which is very useful in practice. Consider $d\theta$ as a form on

$$H_p/C_p.$$

Since it is non-degenerate,

$$\theta \wedge (d\theta)^n \neq 0, \tag{3.9}$$

where $2n = \dim (H_p/C_p)$, while

$$\theta \wedge (d\theta)^{n+1} = 0 \tag{3.10}$$

($(d\theta)^n$ denotes the exterior product

$$d\theta \wedge \ldots \wedge d\theta$$

of n copies of $d\theta$). Now, the definitions given above add up, as far as dimensions go, as follows:

$$m-1 = 2n+r, \tag{3.11}$$

Hence, if n is the integer which satisfies (3.9) and (3.10), then

$$\begin{aligned} n + r &= n + m - 1 - 2n \\ &= m - n - 1. \end{aligned} \tag{3.12}$$

<u>Exercise</u>. Fill in the details of the proof of this method of calculating the dimension of the maximal integral submanifolds.

<u>Example</u>. <u>Simple thermodynamic systems</u>

As in Section 2,

$$M = R^5,$$

with coordinates U, S, T, P, V.

$$\theta = dU - SdT - PdV. \tag{3.13}$$

Then,

$$d\theta = dT \wedge dS + dV \wedge dP.$$

$$(d\theta)^2 = dT \wedge dS \wedge dV \wedge dP$$

$$\theta \wedge (d\theta)^2 = dU \wedge dT \wedge dS \wedge dV \wedge dP \neq 0.$$

Hence,

$$n = 2.$$

The dimension of the maximal integral submanifolds is then

$$5 - 2 - 1 = \underline{\underline{2}} \tag{3.14}$$

Notice that this is indeed the number of "independent variables" chosen in thermodynamics to define the systems.

<u>Exercise</u>. $M = R^{2n+1}$, coordinates

$$(u, x_1, \ldots, x_n, y_1, \ldots, y_n)$$

$$\theta = du - y_1 dx_1 - \ldots - y_n dx_n.$$

Show that the dimension of these maximal integral manifolds is n. Use both methods, i.e. calculate the Cauchy characteristic vectors, and $\theta \wedge (d\theta)^n$.

Now, we shall put these classical mathematical facts to work in the analysis of what is meant by "interaction" in thermodynamics.

4. INTERACTION OF THERMODYNAMIC SYSTEMS

Let M_1, M_2 be manifolds. Suppose θ_1, θ_2 are 1-forms on M_1 and M_2. Let I_1, I_2 be the differential ideal of forms generated by θ_1, θ_2.

(M_1, I_1), (M_2, I_2) may be thought of as thermodynamics presystems. A thermodynamic system associated to (M_1, I_1) is a maximal integral submanifold of I_1.

We ask: What does it mean to let the systems (M_1, I_1), (M_2, I_2) interact? Clearly, this should be some method which assigns to $((M_1, I_1), (M_2, I_2))$ a third system (M_3, I_3), with certain rules. In fact, what is involved here is probably a very general idea, which would be worth pursuing for its own sake. However, at the moment, we pursue only one general version of the examples provided in GPS, Chapter 6.

Recall what was done there. Suppose that:

$$M_1 = R^5 = M_2.$$

Let $(U_1, S_1, T_1, P_1, V_1)$ be coordinates on M_1, $(U_2, S_2, T_2, P_2, V_2)$ on M_2, with the usual physical interpretation for these variables. Set:

$$\theta_1 = dU_1 - S_1 dT_1 - P_1 dV_1 \tag{4.1}$$

$$\theta_2 = dU_2 - S_2 dT_2 - P_2 dV_2 \tag{4.2}$$

$$M = M_1 \times M_2 = R^{10}. \tag{4.3}$$

$U_1, \ldots, V_1, U_2, \ldots, V_2$ are now functions on M. Set:

$$M_3 = \{p \in M: T_1(p) = T_2(p)\} \tag{4.4}$$

$$\theta_3 = \theta_1 + \theta_2 \text{ restricted to } M_3. \tag{4.5}$$

M_3 is the <u>equal temperature</u> or <u>isothermal submanifold</u>.

On M_3, set:

$$T = T_1 = T_2, \tag{4.6}$$

the common temperature. Then, on M_3,

$$\theta_3 = d(U_1 + U_2) - (S_1 + S_2)dT$$
$$- P_1dV_1 - P_2dV_2. \tag{4.7}$$

Set:

$$U = U_1 + U_2; \; S = S_1 + S_2. \tag{4.8}$$

Thus,

$$\theta_3 = dU - SdT - P_1dV_1 - P_2dV_2. \tag{4.9}$$

Set:

$M' = R^7$ = space of variables

$U, S, T, P_1, V_1, P_2, V_2.$

$$\theta' = dU - SdT - P_1dV_1 - P_2dV_2. \tag{4.10}$$

Let

$$\alpha\colon M_3 \to M'$$

be the map such that:

$$\alpha((U_1, S_1, T, P_1, V_1), (U_2, S_2, T, P_2, V_2)) \\ = (U_1 + U_2, S_1 + S_2, T, P_1, V_1, P_2, V_2). \tag{4.11}$$

Here is a basic principle

<u>The result of isothermally interacting two systems at equilibrium defined by</u> (M_1, θ_1), (M_2, θ_2) <u>is an equilibrium system described by</u> (M', θ').

Note that the maximal integral manifolds of θ' are three dimensional. Here is a way to construct them. Suppose given maps:

$$\varphi_1\colon R^2 \to M_1,\ \varphi_1{}^*(\theta_1) = 0 \\ \varphi_2\colon R^2 \to M_2\colon\ \varphi_1{}^*(\theta_2) = 0. \tag{4.12}$$

Let

$$\varphi_3 = \varphi_1 \times \varphi_2\colon R^2 \times R^2 \to M_1 \times M_2 \tag{4.13}$$

be the product map. Consider the function

$$\varphi_3{}^*(T_1 - T_2). \tag{4.14}$$

Let

$$N' = \text{subset of } R^2 \times R^2 \text{ on which} \\ \varphi_3{}^*(T_1 - T_2) = 0. \tag{4.15}$$

Suppose that N' is a 3-dimensional submanifold of $R^2 \times R^2 = R^4$.

Remark: Alternately, N' may be defined as the intersection

$$\varphi_3(R^2 \times R^2) \wedge M'$$

Set:

$$\varphi' = \alpha\varphi_3 : N' \to M'.$$

Notice that:

$$\varphi'^*(\theta') = 0,$$

i.e. the submanifold

$$\varphi'(R^3) \subset M'$$

is the thermodynamic system resulting from isothermally interacting the systems $\varphi_1(R^2)$, $\varphi_2(R^2)$.

This example is clearly a fascinating model of how to define a concept of "interaction" which is both geometric and very general. We now go back to the source of all this intuition - classical mechanics - to understand better how this machinery works. In fact, the general formulation of "classical mechanics" given in GPS, Chapter 2 enables one to present the notion of "mechanical interaction" in a geometric form which is very similiar to the way we have presented "thermodynamic interaction" above. This is the subject of the next chapter.

Chapter X

THE GEOMETRY OF INTERACTION IN CLASSICAL MECHANICS

1. INTRODUCTION

In the previous chapter, we have described a geometric setting for the notion of "interaction" in thermodynamics. We now treat classical mechanics, and show that there the interaction notion may be formulated in a very natural, analogous way.

The best mathematical reference for the classical ideas of mechanics is Whittaker [1]. The modern embellishments can be found in DGCV, LAQM, GPS, and the books by Abraham and Marsden, Godbillon, and Souriau. In fact, E. Cartan already understood this way of looking at mechanics, as is evident from his great book "Lecons sur les invariants integraux," and his papers (particularly those on Relativity) in Part III of his Collected Works.

For the sake of simplicity, we shall mainly work with coordinates (the "configuration-momentum coordinates" of traditional Hamilton-Jacobi theory), than translate the results into manifold terminology, which is in fact a relatively simple task.

2. MECHANICAL SYSTEMS

In Whittaker [1], a "mechanical system of n degrees of freedom" is usually defined by the following data:

> Coordinates (q_i), $1 \le i, j \le n$, called the <u>configuration variables</u>. Coordinates (p_i), called the <u>momentum variables</u>. Together, the coordinates (q_i, p_i) of R^{2n} define the <u>phase space</u>. A <u>time variable</u> t.
>
> A function $h'(p, q, t)$ of these variables, called the <u>Hamiltonian</u>.

This data then defines the time evolution of the system, via the Hamilton equations:

$$\frac{dq_i}{dt} = \frac{\partial h'}{\partial p_i}$$
$$\frac{dp_i}{dt} = - \frac{\partial h'}{\partial q_i}. \qquad (2.1)$$

The link with the manifold-theoretic way of looking at mechanics is that the solutions of (2.1) are identical with the <u>characteristic curves</u> of the following 2-form on R^{2n+1}:

$$\omega' = dp_i \wedge dq_i - dh' \wedge dt. \qquad (2.2)$$

(This basic observation is developed in Cartan's book "Invariants Integraux"; see also DGCV, LAQM and GPS.)

Let us now reinterpret these ideas in the following way. Enlarge the space to R^{2n+2}, the space of variables

$$(p_i, q_i, t, h). \tag{2.3}$$

Thus, we add a new variable h, called the <u>energy variable</u>. On this space, consider the following 2-form:

$$\omega = dp_i \wedge dq_i + dt \wedge dh. \tag{2.4}$$

<u>Remark</u>: Formula (2.4) should suggest to the reader the fundamental idea that the momentum variables p_i are <u>conjugate</u> to the position variables, while the energy variable h is conjugate to the time variable t. Of course, in quantum mechanics such conjugate variables should satisfy the <u>Heisenberg uncertainty relations</u>.

<u>Definition</u>. Let M be R^{2n+2}, the space of variables (p_i, q_i, t, h). A <u>mechanical system</u> is a submanifold M' of M such that:

$$\dim M' = 2n+1. \tag{2.5}$$

The 1-forms

$dp_1, \ldots, dp_n, dq_1, \ldots, dq_n, dt$
are linearly independent when restricted to M'. (2.6)

Let us now see what these conditions mean in terms of local coordinates. (2.5) and (2.6) imply that, locally at least,

$$(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$$

form a coordinate system for M'. Set:

$$h' = h \text{ restricted to } M' \tag{2.7}$$

$$\omega' = \omega \text{ restricted to } M'. \tag{2.8}$$

Then, h' is a function

$$h'(p, q, t)$$

of these variables.

$$\omega' = dp_i \wedge dq_i - dh' \wedge dt. \tag{2.9}$$

Thus formula leads us back then to a "mechanical system" as it is defined by Whittaker.

Here is another way of stating these observations:

<u>Definition</u>. A <u>(classical)</u> <u>mechanical system</u> is a triple

$$(M, M', \omega)$$

consisting of an even dimensional manifold M, a maximal rank closed 2-form ω on M (so that M is what is known as a <u>symplectic</u> or <u>canonical</u> manifold), and a submanifold M' of M of codimension 1, such that:

At each point of M', the characteristic tangent vectors of ω restricted to M' are 1-dimensional. (2.10)

The relation defining M' is called the constitutive relation of the mechanical system.

Remark: One way of defining such an M is to let it be the cotangent bundle $T^d(N)$ to a manifold N of dimension $(n+1)$. In terms of the local coordinates, N is the space of variables (q, t), i.e. the "space-time" manifold.

Definition. Let (M, M', ω) be a mechanical system. A curve in M' is said to be an orbit of the system if it is an integral curve of a vector field $X \in V(M')$ such that

$$X \lrcorner \omega' = 0,$$

i.e. if the tangent vector to the curve at each point is a characteristic vector of ω'.

Often, group-theoretical considerations serve to pick out interesting constitutive relations. Here are some relevant ideas.

3. ISOMORPHISM AND SYMMETRY GROUPS OF MECHANICAL SYSTEMS RELATIVISTIC AND NON-RELATIVISTIC MECHANICS

Let M be an even dimensional manifold, with a closed 2-form ω of maximal rank on M. Let G be a group of diffeomorphisms acting on M.

Definition. G is said to act as a group of canonical transformations on M if:

$$g^*(\omega) = \omega \tag{3.1}$$

for all $g \in G$.

Definition. Let (M, M', ω) be a mechanical system, as defined in Section 2. Let G be a transformation group on M. G is said to be an isomorphism group for the system if each $g \in G$ satisfies (3.1), and also:

$$g(M') = M'. \tag{3.2}$$

If (3.1) and (3.2) are satisfied, we see that g maps an orbit curve of the system into another orbit curve, i.e. g is a symmetry of the system, in the usual sense.

Example 1. The free relativistic particle and its non-relativistic limit.

Let

$N = R^4$, the space-time manifold of special relativity.

$$M = T^d(N), \text{ the cotangent bundle to } N.$$

Let (x_μ), $0 \le \mu, \nu \le 3$, be the coordinates of N, labelled as usual in relativistic physics, i.e.

$$x_0 = \text{time variable}$$
$$x_i,\ 1 \le i, j \le 3, = \text{space variables.}$$

Let (p_μ) be the momentum variables on $M = T(N)$, i.e.

$$p_\mu(\theta) = \theta(\frac{\partial}{\partial x_\mu})$$

Let $(g_{\mu\nu})$ be the Lorentz metric tensor, i.e.

$$g_{11} = -1 = -g_{22} = -g_{33}$$
$$g_{00} = \frac{1}{c^2}$$
$$g_{01} = 0 = \text{etc.}$$

(c is the velocity of light).

Set:

$$L_c = g_{\mu\nu}p_\mu p_\nu$$

M_c' = points of M on which

$$L_c = c^2. \tag{3.3}$$

$$h = p_{0i}p_i = p_i$$

$$g_i = x_{ii}t = x_0.$$

$$\omega = dp_\mu \wedge dx_\mu$$
$$= dp_i \wedge dq_i + dh \wedge dt.$$

On M', we have:

$$h^2 = c^2(f_i p_i + c^2),$$

or

$$h_c' = c\sqrt{p_i p_i + c^2} \tag{3.4}$$

It will be recognized that h_c' given by (3.3), is the Hamiltonian of the free relativistic particle.

Exercise. Show that the solutions of the 3-dimensional Hamilton equations, with Hamiltonian (3.3), are the same as the solutions of the Euler-Lagrange equations, with Lagrangian

$$L' = c\sqrt{c^2 - \dot{x}_1^2 - \dot{x}_2^2 - \dot{x}_3^2}.$$

With this formalism, we can see what happens as $c \to \infty$, i.e. in the non-relativistic limit. To do this, we must introduce new variables. Introduce a "vectorial" notation:

$$\vec{p} = (p_i).$$

Then, (3.3) takes the form:

$$p_0{}^2 - c^2\vec{p}^2 = c^4. \tag{3.5}$$

Set:

$$p_0' = \frac{1}{2}\left(\frac{p_0{}^2}{c^2} - c^2\right). \tag{3.6}$$

Then, (3.5) takes the form:

$$(c^2 + 2p_0')c^2 - c^2\vec{p}^2 = c^4$$

or

$$p_0' = \frac{1}{2}\vec{p}^2. \tag{3.7}$$

<u>Remark</u>. Notice that this is the formula for the <u>non-relativistic</u> energy of the free particle. The interpretation of relation (3.7) is that it parameterizes the submanifold M_c' of M.

We must compute ω and ω' restricted to M_c'. From (3.6),

$$\begin{aligned} dp_0' &= \frac{p_0 dp_0}{c^2} \\ &= \frac{c\sqrt{2p_0' + c^2}}{c^2}\, dp_0 \\ &\to dp_0 \text{ as } c \to \infty. \end{aligned}$$

Thus, as $c \to \infty$, the differential form:

$$\omega = dp_i \wedge dx_i + dp_0 \wedge dt$$

goes into the differential form:

$$\omega_\infty = dp_i \wedge dx_i + dp_0' \wedge dt.$$

In particular, the characteristic curves of

$$\omega' = dp_i \wedge dx_i + d(c\sqrt{\vec{p}^2 + c}) \wedge dt$$

go over, as $c \to \infty$, into the solution curves of

$$\omega'_\infty = dp_i \wedge dx_i + d(\tfrac{1}{2} p^2) \wedge dt.$$

This is the precise "geometric" form of the statement that Newtonian mechanics is the "limit" as $c \to \infty$ of relativistic mechanics.

Remarks: Another geometric approach - based on a "dual" calculus of variations approach - is described in DGCV, Chapter 16.

There is a considerable amount of work to do to clarify the precise meaning of this example. From a more general point of view, we have a typical geometric "deformation" situation, which should be (and will be, in a later Volume) investigated further. Let us review, from global point of view, what has been done. M is the cotangent bundle to $N = R^4$. As a cotangent bundle, M has a closed two-form ω, which defines it as a canonical manifold. (See Abraham and Marsden [1], and VB, vol. II.) M_c' is a submanifold of M of codimension one. The characteristic curves of ω restricted to M_c - parameterized by $t \equiv x_0$ - are the curves of interest. Now, M_c' has no limit as $c \to \infty$ in M. One can either give it a limit by compactifying M (e.g., by embedding each fiber of $M = T^d(N)$ as an open affine variety of a projective space), or constructing, for each real c a differ-morphism

$$\varphi_c : M \to M$$

such that:

$$\lim_{c\to\infty} \varphi_c(M_c') \text{ exists.}$$

We have essentially chosen the latter course: φ_c is defined by formula (3.6). What we showed is that:

$$\lim_{c\to\infty} \varphi_c{}^*(\omega) = \omega,$$

i.e. that

φ_c is <u>asymptotically</u> a canonical transformation.

In itself, this might be an interesting new geometric concept, particularly since the limit, as $c \to \infty$, of φ_c itself does not exist.

Another interesting topic, implicit in this simple example, is the deformation of the group of automorphisms of the system for finite c, i.e. the deformation of Poincaré group, into the Galilean group. This topic has been briefly treated in my series of papers, "Analytic continuation of group representations." A topic of prime <u>algebraic</u> interest is that this limiting process is such that "true" representations of the Poincaré group have as limit "ray" or "projective" representation of the Galilean group, i.e. "true" representations of a "central extension" of the Galilean group. Presumably, this algebraic property is related to the singular nature of the geometric deformation problem, described above. Again, there is much work to be done to make a consistent, complete theory - which can then presumably be applied to more complicated and interesting geometric, physical and algebraic situations.

Here is another direction in which one can generalize.

Exercise. Relativistic mechanics with external potentials

Let $A_\mu(x)$, $0 \le \mu \le 3$, be four functions of space-time points: (Physically, they may represent the potential of an electromagnetic field). Replace the "free" function L_c defined above by the following one; using the usual "gauge-invariant" rules:

$$L_c = g_{\mu\nu}(p_\mu - A_\mu)(p_\nu - A_\nu). \qquad (3.8)$$

Carry out the derivation of the differential equations - in standard Hamiltonian form - of the corresponding mechanical system, and describe the geometric limiting process as $c \to \infty$.

Example 2. Newtonian physics for a single particle. Covariance under the Galilean group

We shall again start off with $N = R^4$, the space-time manifold of ordinary physics. Set:

$$M = T^d(N),$$

the cotangent bundle to N. Let

$$(t,\ x_i),\ 1 \le i,\ j \le 3,$$

be coordinates on N, as usual, t is the time coordinate, and (x_i) are the space variables. Let

$$(h,\ p_i)$$

be the corresponding "momentum" functions on $T^d(M)$, i.e.

$$h(\theta) = \theta(\frac{\partial}{\partial t}) \tag{3.9}$$

for $\theta \in T^d(N)$.

Recollection of terminology: θ is a covector to N. By definition, this means that θ is a linear combination of dt and the dx_i. $\frac{\partial}{\partial t}$ is a vector field on N. This explains how the right hand side of (3.9) is evaluated. For example, if

$\theta = a\ dt$, then

$h(\theta) = a.$

Similiarly,

$$p_i(\theta) = \theta(\frac{\partial}{\partial x_i}). \tag{3.10}$$

Let:

$$\Theta = dp_i \wedge dx_i + dh \wedge dt. \tag{3.11}$$

define a submanifold M' of M via the following relations:

$$-h = \frac{1}{2m} p,\ p_i + V(x,\ t) + p_i a_i(x,\ t) \tag{3.12}$$

V and the a_i are functions on N, i.e. functions of space-time points. (V is called a <u>scalar potential</u>, while (a_i) defines the <u>vector potential</u>). m is a real parameter, the <u>mass</u> of the particle. Comparing (3.11) and (3.12), we see that the characteristic curves of ω restricted to M' are precisely the solution of the Hamilton equation, with the right hand side of (3.12) as the Hamiltonian. Here is an exercise which explains the role of the <u>Galilean group</u>.

<u>Exercise</u>. Let φ: N → N be a diffeomorphism of space-time. Let

$$\varphi_*: T(N) \to T(N)$$

be the prolongation of φ to the tangent bundle.

For each p ∈ N, φ_* maps

$$N_p \to N_{\varphi(p)}.$$

Hence,

$$\varphi_*^d \text{ maps: } N_{\varphi(p)}^d \to N_p^d.$$

Replacing φ with φ^{-1}, we see that

$$(\varphi^{-1})_*^d \text{ maps } N_p^d \to N_{\varphi(p)}^d$$

as p varies, we obtain a bundle isomorphism

$$\varphi': T^d(N) = M \to T^d(N) = M$$

called the <u>covector prolongation</u> of φ. Determine the conditions φ must satisfy in order that φ' map a submanifold M' of type

(3.12) onto another of the <u>same type</u>, determined say by relations of the form:

$$h = \frac{1}{2m'} p_i p_i + V'(x, t) + p_i a_i'(x, t). \qquad (3.13)$$

Show that φ must belong to a certain subgroup of the group of affine automorphisms of the vector space $N = R^4$. This subgroup is called the <u>Galilean group</u>. Finally, determine the relation of the functions V', a_i' in (3.13) to the V, a_i in (3.12). Show that:

$$V' = \varphi^{-1*}(V). \qquad (3.14)$$

$$a_i' dx_i = \varphi^{-1*}(a_i dx_i). \qquad (3.15)$$

The physicists would say that (3.14) is a <u>scalar</u> transformation law, while (3.15) is a <u>vectorial</u> transformation law. Finally show that the numbers M and M' appearing in (3.12) and (3.13) must be equal, i.e. "mass" is invariant under Galilean transformation.

4. INTERACTIONS OF MECHANICAL SYSTEMS

The reader should now be aware of the analogy of the way we have defined a "mechanical system" in this chapter, and the way a "thermodynamic system" was defined in the previous chapter. The common theme is, of course, that the two notions are defined in terms of the same sort of mathematics, i.e. the theory of exterior differential systems. Having understood the notion of "interaction" in thermodynamics (more precisely, the simplest such notion; the more complicated "non-equilibrium" situations are still enveloped in some sort of quasi-mathematical fog that I do not understand too well) let us develop the analogous mathematical notions of "interaction" for mechanical systems, and see whether they make physical sense.

First, consider the case of the interaction of two mechanical systems. Suppose for simplicity, that each system has one degree of freedom. Let q be the configuration variable, p the momentum variable of the first particle, (q', p') the configuration-momentum variables of the second particle. Introduce

$$(h, t)$$

as the conjugate energy-time variables of the first system,

$$(h', t')$$

as the energy-time variables of the second system. Let

M = space of variables

$$(p, q, h, t)$$

M' = space of variables

$$(p', q', h', t').$$

M and M' may be called the extended phase spaces of the two systems. M and M' are symplectic manifolds, i.e. they are even dimensional, and come equiped with closed, maximal rank 2-forms ω, ω', defined by the following formulas:

$$\begin{aligned} \omega &= dp \wedge dq + dh \wedge dt \\ \omega' &= dp' \wedge dq' + dh' \wedge dt'. \end{aligned} \tag{4.1}$$

Suppose given Hamiltonian functions $H(p, q, t)$, $H'(p', q', t')$ for the time systems. (Note that we have changed notations from Section 2). The relations

$$h = H$$
$$h' = H'$$

determine codimension one submanifolds (geometrically called hypersurfaces) of M, M'. The characteristic curves of ω and ω' restricted to these submanifolds are the trajectories of the mechanical systems determined by

$$(M, \omega, H)$$
$$(M', \omega', H').$$

So far, we have been considering the systems separately. Now, let them "interact." We can do this as follows: Set

$$M \times M' = \text{space of variables}$$
$$(p, q, h, t, p', q', h', t')$$

$$\begin{aligned}\omega + \omega' \\ &= dp \wedge dq + dp' \wedge dq' \\ &+ dh \wedge dt + dh' \wedge dt'\end{aligned}$$

Let M" be the space of variables

$$(p, q, p', q', h'', t'').$$

Set:

$$\omega'' = dp \wedge dq + dp' \wedge dq' + dh' \wedge dt. \tag{4.2}$$

Let

$(M \times M')_{t=t'}$ be the set of points of $M \times M'$ defined by setting $t = t'$.

Define a map

$$\varphi\colon (M \times M')_{\overrightarrow{t=t_1}} M''$$

by the following formula:

$$\begin{aligned}&\varphi((p, q, h, t), (p', q', h', t) \\ &\quad = (p, q, p', q', h + h', t).\end{aligned} \tag{4.3}$$

Comparing (4.2) and (4.3), we have:

$$\varphi^*(\omega'') = \omega + \omega'. \tag{4.4}$$

<u>Definition</u>. The mechanical system defined by (M'', ω''), together with the map φ defined by (4.3), is the mechanical system obtained by <u>interacting</u>, <u>with equal-time relations</u>,

the systems (M, ω), (M', ω').

Thus, a Hamiltonian for the interacted system is a function $H''(p, q, p', q', t)$, determining a hypersurface of the extended phase space M''. The trajectories for the interacted system, determined by H'', are the characteristic curves of the form " restricted to submanifold

$$h'' = H''$$

of M''. In other words, the solutions of the following set of Hamilton equations:

$$\begin{aligned} \frac{dp}{dt} &= \frac{\partial H''}{\partial q}; \quad \frac{dp'}{dt} = \frac{\partial H'}{\partial q'} \\ \frac{dq}{dt} &= -\frac{\partial H''}{\partial p}; \quad \frac{dq'}{dt} = -\frac{\partial H''}{\partial q} \end{aligned} \tag{4.5}$$

Of course, one possible choice for the interaction Hamiltonian H'' is as follows:

$$H'' = H + H'. \tag{4.6}$$

Exercise. With the choice of (7.6) of interaction Hamiltonian, show that the trajectories are the Cartesian product of the trajectories of the individual systems (M, ω), (M', ω'). Is the converse true? This sort of interaction is called a free or no-interaction interaction.

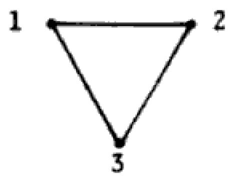
Remark: A) Once having seen things in this way, there is nothing particularly sacred about the choice of <u>time</u> as the variable to equate to define "interactions." Any pair of functions on M and M' would do! In practice, the choice is usually determined by appropriate "symmetry" consideration, i.e. one is given a group G which acts as canonical automorphisms on M and M', then lets G act via the diagonal action as $M \times M'$, and chooses functions $f: M \times M' \to R$ so that this diagonal action leaves invariant the relation

$$f = 0.$$

For G = the Galilean group, the function $f = t - t'$ is the appropriate one, - indeed, this is built into the physical role that "time" plays in Newtonian physics. For G = Poincaré group, there is no such obvious choice. This ambiguity in the possible way of defining "direct two particle interaction" in <u>relativistic mechanics</u> makes it a less interesting physical theory than the corresponding Newtonian case. (Although there is a interesting and extensive literature on this - see Künzle [1]). Indeed, I believe that the conventional wisdom among physicists is that it is necessary to have a <u>field theory</u> to successfully define "interaction" in relativistic physics.

B) There is also an interesting "combinatorics" of interactions. This plays no great role in the physics literature, so far at least, but is very important in chemistry, engineering, economics, etc. For example, we could symbolize the simplest sort of "interaction" considered in this section by the following graph:

The two vertices correspond to the component systems (M, ω), (M', ω'). To see what "interaction" would mean in more complicated combinatoric situations, consider the next most complicated graph, a triangle:



This represents three systems,

$$(M_1, \omega_1), (M_2, \omega_2), (M_3, \omega_3).$$

Set:

$$M = (M_1 \times M_2 \times M_3, \omega = \omega_1 + \omega_2 + \omega_3).$$

Let

$$f_{12}: M_1 \times M_2 \to R$$

$$f_{13}: M_1 \times M_3 \to R$$

$$f_{23}: M_2 \times M_3 \to R$$

be real-valued functions, associated with the lines in the above graph. Set:

M' = submanifold of M defined by relations

$$f_{12} = 0 = f_{13} = f_{23}.$$

We touch here on relations between network-graph theory and mechanics which I intend to develop in greater detail later on.

5. THE GENERAL THEORY OF INTERACTIONS AND THE KRON-HOFFMAN METHODS

The engineer Gabriel Kron developed an extensive theory of "systems" with a strong geometric flavor, tied to the mathematics of classical tensor analysis. Apparently, Kron did not receive adequate recognitoin for his ideas, although now they can be seen more clearly as a precursor of modern "systems theory." Banesh Hoffman was one of the few people familiar with the physical and mathematical background to Kron's work to appreciate it, and he published an important paper [1] showing how Kron's ideas worked in terms of simple classical mechanical systems. I will now take up the material in Hoffman's paper, in terms of the general notions of "interactions" in development here. I am indebted to

W. Jerkovsky of the Aerospace Corporation for correspondence about this material, and for drawing my attention to Hoffman's paper. He has also provided me with copies of extensive manuscripts (in preparation for publication) which extend the Kron-Hoffman methods to systems which occur in modern aerospace technology.

Here is a simple example which illustrates the Kron method. Consider two particles, with coordinates x_1, x_2, masses m_1, m_2, moving on a line, with force law f_1, f_2. Newton's equations are

$$\begin{aligned} m_1 \frac{d^2x_1}{dt^2} &= f_1 \\ m_2 \frac{d^2x_2}{dt^2} &= f_2. \end{aligned} \tag{5.1}$$

The result of introducing a constraint

$$x_1 = x_2 = x$$

is to derive a <u>new</u> system

$$(m_1 + m_2) \frac{d^2x}{dt^2} = (f_1 + f_2). \tag{5.2}$$

The question is: How is (5.2) obtained from (5.1)? Notice that it is not obtained in the obvious mathematical way by saying that for every solution $t \to x(t)$ of (5.2), the curve

$$t \to (x(t) = x_1(t), x(t) = x_2(t))$$

in R^2 should be a solution of (5.1).

One systematic way is to rewrite equations (5.1) in terms of "Lagrangians" and "force-law forms", as described in DGCV. Let

$$M = R^4 = \text{space of variables}$$

$$x_1,\ x_2,\ \dot{x}_1,\ \dot{x}_2$$

$$L = \frac{1}{2} m_1 \dot{x}_1^{\,2} + \frac{1}{2} m_2 \dot{x}_2^{\,2} \tag{5.3}$$

$$= \text{Lagrangian of "free" particles } x_1,\ x_2.$$

Let θ be the Cartan form of L, i.e.

$$\theta = L\,dt + \frac{\partial L}{\partial \dot{x}_1}(dx_1 - \dot{x}_1 dt + \frac{\partial L}{\partial \dot{x}_2}(dx_2 - \dot{x}_2 dt)$$

$$= m_1 \dot{x}_1 dx_1 + m_2 \dot{x}_2 dx_2 - H\,dt. \tag{5.4}$$

Let $t \to (x_1(t), \dot{x}_1(t), x_2(t), \dot{x}_2(t) = \sigma(t)$ be a curve in M.

<u>Warning</u>. $\dot{x}_1(t)$ is not <u>necessarily</u> the time derivative of $x_1(t)$. At the moment, x_1 is just another variable. (Differential-geometrically, it is the linear coordinate of the tangent bundle to the manifold where coordinate is x_1).

<u>Exercise</u>. Show that $t \to (x_1(t), x_2(t))$ are solutions of Newton's equations (5.1), with:

$$\dot{x}_1(t) = \frac{dx_1}{dt},\ \dot{x}_2(t) = \frac{dx_2}{dt},$$

if and only if:

$$\sigma'(t) \lrcorner\, d\theta = f_1 dx_1 + f_2 dx_2 \tag{5.5}$$

($\sigma'(t) \in M_{\sigma(t)}$ is the tangent vector to the curve). (As I have shown in GPS, equations (5.5) are the ultimate coordinate-free version of Newton's laws. The key fact from our point of view is that they are expressed in terms of <u>differential forms</u>, which of course have a "tensorial" transformation law under arbitrary mappings, <u>not</u> just under diffeomorphisms, as do vector fields.)

Now, let M' be R^2, the space of variables $(y, \dot{y})$. Constrain the mapping

$$\varphi\colon M' \to M$$

as follows:

$$\varphi(y, \dot{y}) = (y, \dot{y}, y, \dot{y}). \tag{5.6}$$

<u>Remark</u>: In other words, $\varphi(M')$ is the submanifold of M defined by the following relations: $x_1 = x_2$, $\dot{x}_1 = \dot{x}_2$. If one interprets M as the tangent bundle to N, the space of variables (x_1, x_2), then $\varphi(M')$ is the tangent bundle to the submanifold $:x_1 = x_2:$ of N.

Call the differential form on the right hand side of (5.2) the <u>force law form</u>, denoted by η.

Remark: This is a special case of the general principle, described in GPS that:

$$X \lrcorner \,(\text{Cartan form}) = \text{Force law form}, \tag{5.7}$$

where X is the vector field whose integral curves are the trajectories of the dynamical system. It is writing the system (5.1) in this form which enables us to introduce "constraints" or "interactions" in a reasonable way.

Set:

$$\theta' = \varphi^*(\theta) = \theta \text{ restricted to the submanifold } \varphi(M').$$

$$\eta' = \varphi^*(\eta).$$

Exercise. Let

$$L' = \varphi^*(L) = \frac{1}{2}(m_1 + m_2)\dot{y}^2. \tag{5.8}$$

Show that:

$$\theta' = \text{Cartan form of the Lagrangian } L'. \tag{5.9}$$

Remark: (5.9) is a general property of the assignment (or "functor") - (Lagrangian) → (Cartan form). Also,

$$\eta' = (\varphi^*(f_1) + \varphi^*(f_2))dy. \tag{5.10}$$

We see now that the equations of curves $t \to \alpha(t)$ in M' which satisfy

$$\alpha'(t) \lrcorner \theta' = \eta,$$

are precisely the solutions of (5.2). Thus, system (5.2) has been obtained from (5.1) by this "constraint" process. A key feature of this process was the conversion of the original "vector-field" form of the equation of motion, namely (5.1), into a "dual, differential-form" version, since differential forms always pull-back under mappings, whereas vector fields do not transform in a reasonable way under arbitrary mappings.

Exercise. Go through Hoffman's exposition [1] of Kron's ideas and show that the other examples considered there may be put into the same general framework, i.e. one starts with two "primitive" mechanical systems, defined by means of differential forms on manifolds

$$M_1, M_2.$$

(In the above example, $M_1 = \{(x_1, \dot{x})\}$, $M_2 = \{(x_2, \dot{x}_2)\}$.) Form the "free" composite system

$$M = M_1 \times M_2.$$

Introduce a constraint submanifold

$$M' \subset M,$$

restrict the differential forms to M', which are then used to define the "interacting" system on M'. One thinks of M' as built up from M_1, M_2 by "tearing."

I will now formulate some general ideas.

6. A GENERAL DEFINITION OF CONSTRAINT-INTERACTION FOR MECHANICAL SYSTEMS

First, we need a definition of "mechanical system."

Definition. A mechanical system is defined by a quadruple

$$(M, t, \theta, \eta),$$

where:

> M is a manifold t
>
> t is a real-valued, C^∞ function on M, called the time function. θ, η are 1-forms. θ is called the Cartan form, η the force-law form.

The trajectories of the system are the curves in M which are integral curves of the vector fields $X \in V(M)$ such that:

$$\begin{aligned} X \lrcorner\, d\theta &= \eta \\ X(t) &= 1. \end{aligned} \tag{6.1}$$

Remark: This is the appropriate definition for "Newtonian" physics, where there is an "absolute time function." The techniques can be changed to cover relativistic situations also.

Now, we can put together two mechanical systems, in a "free" or "no interacting" way. Let

$$(M_1, \theta_1, \eta_1, t_1), (M_2, \theta_2, \eta_2, t_2)$$

be two systems. Construct a system as (M', θ', η', t) as follows

M' = set of all points of $M_1 \times M_2$ at which $t_1 = t_2$.

$\theta' = \theta_1 + \theta_2$ considered as a form on $M_1 \times M_2$, restricted to M'.

$\eta' = \eta_1 + \eta_2$, considered as a form on $M_1 \times M_2$, restricted to M'.

(M', θ', η', t), defined in this way, is called the <u>free composite system</u>, defined by the subsystems $(M_1, \theta_1, \eta_1, t_1)$, $(M_2, \theta_2, \eta_2, t_2)$. The construction corresponds to the obvious physical idea, in case they are "particles", of putting them together, as in the following exercise.

<u>Exercise</u>. Write out explicitly and formally (informally, it was done in Section 7) the free composite corresponding to two particles moving in R^3 according to Newton's laws:

$$M_1 \frac{d^2\vec{x}_1}{dt^2} = \vec{f}_1(\dot{\vec{x}}_1, \vec{x}_1, t)$$

$$M_2 \frac{d^2\vec{x}_2}{dt^2} = \vec{f}_2(\dot{\vec{x}}_2, \vec{x}_2, t)$$

Definition. Let (M', θ', η', t') be the free composite built up from $(M_1, \theta_1, \eta_1, t_1)$, $(M_2, \theta_2, \eta_2, t_2)$. Let M be a manifold,

$$\varphi: M \to M'$$

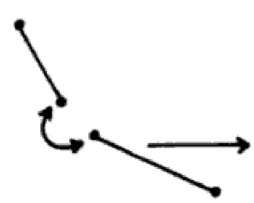
a mapping. Set:

$$\theta = \varphi^*(\theta), \ \eta' = \varphi^*(\theta), \ t = \varphi^*(t').$$

Then, the mechanical system (M, θ, η, t) is the system resulting from the "primitive" systems $(M_1, \theta_1, \eta_1, t_1)$, $(M_2, \theta_2, \eta_2, t_2)$, using the "constraints" φ. (In typical situations, e.g. the Kron-Hoffman example treated in Section 7, φ will be a submanifold mapping.)

Remark and exercise.

A class of examples particularly well suited to this sort of interaction are "linkages." Hoffman's exposition [1] of Kron's ideas contains an extensive discussion. For example:



The upper link is fixed at its upper vertex. The lower one is free to move, subject to an external force $\longrightarrow$. The "constraint," denoted by , links them together.

Exercise. Using the kinematic formulas in Hoffman's paper, set up the equations for the interacting system in terms of the general formalism of this paper.

Final remarks: Let us emphasize again that it is the differential-geometric properties of differential forms that make them so useful in setting up constraints and interactions. The *equations* of mechanics are given in terms of vector fields, which do not transform, except under diffeomorphisms. However, the "kinematics" is given in terms of *differential forms*. I will now describe a framework that is common to many examples of thermodynamics *and* mechanics, involving what are called *Cauchy characteristic vector fields*. (The terminology and idea are due to Cartan. See GPS for details.)

7. INTERACTIONS AND CAUCHY CHARACTERISTICS

I shall now briefly describe a general differential-geometric framework within which one can think of our work on "interactions" for *both* classical mechanics and thermodynamics. (I believe that "interactions" for fields and quantum systems may be described in an analogous way, but at the moment this remains in the background).

Let M be a manifold, and let

$$I$$

be a <u>differential ideal</u> of differential forms. Recall (e.g. from GPS) that this means that:

$$I + I \subset I$$

$$F^r(M) \wedge I \subset I$$

for all r.

$$dI \subset I. \tag{7.1}$$

<u>Definition</u>. A <u>Cauchy characteristic vector field</u> for I is an $X \in V(M)$ such that:

$$X \lrcorner I \subset I. \tag{7.2}$$

C(I) denotes the set of these vector fields. C(I) is <u>completely integrable</u>, i.e.

$$[C(I), C(I)] \subset C(I). \tag{7.3}$$

<u>Remark</u>: Condition (7.3) is a consequence of (7.1).

<u>Definition</u>. Let M, M' be manifolds, I and I' differential ideals of differential forms on M and M'. A map

$$\varphi: M \to M'$$

is said to be a <u>Cauchy homomorphism</u> of (M, I) to (M, I') if:

$$\varphi^*(I') \subset I. \tag{7.4}$$

Now, let

$$(M_1, I_1)$$

$$(M_2, I_2)$$

be two differential ideals on different manifolds I_1, I_2.

We can now form a third system (M, I) as follows:

$$M = M_1 \times M_2 \tag{7.5}$$

$$I = \text{differential ideal} \tag{7.6}$$

of forms generated by

$$\pi_1{}^*(I_1) + \pi_2{}^*(I_2),$$

where $\pi_1: M \to M_1$, $\pi_2: M \to M_2$

are Cartesian product projection maps.

(M, I) is the <u>no-interaction composite</u> of (M_1, I_1) and (M_2, I_2).

Let $M' \subset M$ be a submanifold of $M = M_1 \times M_2$. Let

$$I' = I \text{ restricted to } M'.$$

<u>Definition</u>. A differential ideal system (M", I"), together with a Cauchy homomorphism

$$\varphi:(M', I') \to (M'', I''),$$

is said to be the result of a interacting (M_1, I_1) and (M_2, I_2), under constraints M'.

That is about all I will say in general about this set-up. There is an obvious program to follow up, and "classify" the possible types of interactions. For example, one might be given a group G of automorphisms of (M_1, I_1) and (M_2, I_2), let G act as the diagonal action

$$M_1 \times M_2,$$

and require that M' be invariant under this action of G. This is the way "Lorentz invariant" and "Galilean invariant" particle interactions are introduced into classical mechanics.

To fix the ideas more firmly, I give again the two main examples, from thermodynamics and mechanics.

Example A. Mechanics

Consider two mechanical systems; for simplicity, suppose both have one degree of freedom. Let

$$(q_1, p_1), (q_2, p_2)$$

be the position-momentum variables

$$(h_1, t_1), (h_2, t_2)$$

the energy-time variables.

Let

M_1 = space of variables (q_1, p_1, h_1, t_1)

M_2 = space of variables (q_2, p_2, h_2, t_2)

$\omega_1 = dp_1 \wedge dq_1 + dh_1 \wedge dt_1$

$\omega_2 = dp_2 \wedge dq_2 + dh_2 \wedge dt_2$

I_1 = ideal generated by ω_1

I_2 = ideal generated by ω_2.

$M = M_1 \times M_2$

$\omega = \omega_1 + \omega_2$

M' = set of points of M at which

$t_1 = t_2t.$

Set:

$t \equiv t_1 = t_2$ on M'.

ω' = ω restricted to M'

$= dp_1 \wedge dq_1 + dp_2 \wedge dq_2$

$+ d(h_1 + h_2) \wedge dt.$

Let:

M'' = space of variables

$(p_1, q_2, p_2, q_2, h, t).$

Map φ: $M'' \to M$ as follows

$$((p_1, q_1, h_1, t_1), (p_2, q_2, h_2, t_2))$$
$$= (p_1, q_1, p_2, p_2, h_1 + h_2 = h_1 t = t_1 = t_2)$$
$$\omega'' = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dh \wedge dt.$$

Note that:

$$\varphi^*(\omega'') = \omega_1 + \omega_2,$$

i.e. φ is a Cauchy homomorphism. (M'', ω'') is the composite system, resulting from a "equal time" interaction between (M_1, ω_1) and (M_2, ω_2). Of course, to define a _specific_ interaction, one must provide a _constitutive relation_, i.e. a function

$$h(p_1, q_1, p_2, q_2, t).$$

The graph of this function defines in hypersurface in M''. The _trajectories_ of the system are then the Cauchy characteristic curves of the form ω' restricted to M''.

Example B. _Thermodynamics_

$$M_1 = \{(U_1, P_1, V_1, S_1, T_1)\}$$
$$M_2 = \{(U_2, P_2, V_2, S_2, T_2)\}$$
$$\theta_1 = dU_1 - T_1 dS_1 - P_1 dV_1$$
$$\theta_2 = dU_2 - T_2 dS_2 - P_2 dV_2$$
$$M' = \text{set of points of } M_1 \times M_2 \text{ on which } T_1 = T_2.$$

(at least, this is the appropriate constraint submanifold for "equal temperature," i.e. "isothermal," interactions. See GPS and the previous chapter). Set $T = T_1 = T_2$ = the common temperature. On M',

$$\theta = \theta_1 + \theta_2 = d(U_1 + U_2) - Td(S_1 + S_2)$$
$$- P_1 dV_1 - P_2 dV_2.$$

$$M'' = \{(P_1, V_1, P_2, V_2, U, T, S)\}$$

φ: M' → M'' is defined as:

$$\varphi((U_1, T, S_1, P_1, V_1), (U_2, T, S_2, P_2, V_2))$$
$$= (U_1 + U_2 = U, T, S_1 + S_2, = S, P_1, V_1, P_2, V_2)$$

$$\theta'' = dU - TdS - P_1 dV_1 - P_2 dV_2.$$

Chapter XI

DIFFERENTIAL-GEOMETRIC INTERCONNECTION OF SYSTEMS AND ELECTRIC CIRCUITS

1. INTRODUCTION

The aim of this chapter is to develop a differential-geometric way of defining and studying the notion of "interaction" and "interconnection" of "systems." These ideas are very general, and will apply also to the sort of "systems" that arise in physics, economics, biology, computer science, etc.; however, for the sake of simplicity I will deal here mainly with "systems" as they are known in optimal control theory, where the theory is highly developed. Again, my three basic references for this sort of "system theory" are Brockett [1], Desoer [1], Kalman, Arbib and Falb [1]. Rosenbrock's book [1] contains many interesting algebraic ideas which apply to systems which are more complicated than those which are considered in the first three references. Finally, one should note that the work of G. Kron in the 1930's and 40's is an important predecessor of much of what I am doing, although at the moment I do not make direct use of it. (Indeed, I do not understand it very well, beyond an extremely illuminating exposition by B. Hoffman [1], which was pointed out to me by W. Jerkovsky of Aerospace Corporation. I am indebted to M. Jerkowsky for correspondence about the Kron ideas.)

The approach to systems theory developed in the works mentioned above might be thought of mathematically as the "theory" of physical systems "modelled" by ordinary differential equations, linear or non-linear. (Of course, one can usually only handle things precisely in the linear case, but one should not forget that the world is non-linear. Economics, particularly, requires a "non-linear systems theory," of a sort that does not yet exist.) Thus, this sort of "systems theory" is related to the mathematical discipline which we think of as "differential equations" as "metamathematics" is related to "mathematics." (One can of course approach it without mentioning a "differential equation" - and many people do - but I believe that doing this cuts one off from a major source of intuition, particularly in terms of geometry and physics.)

Part of my interest in this chapter will be to show how the ideas may be cast into a unified geometric form using the ideas of E. Cartan's theory of exterior differential systems. (See my book, "Geometry, physics and systems.") In previous chapters I have already dealt with the notion of "interaction" in mechanics and thermodynamics in this context, so the reader will realize that we are working towards a formalism which will unify wide areas of science.

2. SYSTEMS MODELLED BY ORDINARY DIFFERENTIAL EQUATIONS ON FINITE DIMENSIONAL VECTOR SPACES

Let U, X, Y be vector spaces. We shall suppose for simplicity that they are finite dimensional, real, vector spaces, although generalizations in various mathematical directions may readily be considered. Here are some examples of such direction:

a) Infinite dimensional vector spaces.

Functional analysis becomes prominent here.

b) Finite dimensional manifolds.

This involves differential geometry, particularly the theory of exterior differential systems. This will be discussed later on in this treatise.

c) Infinite dimensional manifolds.

A combination of a) and b). Involves what is known as "Global analysis."

d) Vector spaces over arbitrary fields.

Differential equations are replaced by difference equations. Relevant to coding and communication theory, computer science, etc.

e) Algebraic varieties.

Utilizes algebraic geometry. See Volume VIII.

Putting these general possibilities to the side, at least for the moment, we shall work in the context (or "category") of finite dimensional real vector spaces. U is called the input space, X the state space, and Y the output space. The system equations are ordinary differential equations of the following form.

$$\frac{dx}{dt} = f(x, u, t) \tag{2.1}$$

$$y = g(x, u, t). \tag{2.2}$$

t denotes a real variable,

$$-\infty < t < \infty.$$

f, g are maps, with domain and range as follows:

$$f: X \times U \times R \to X \tag{2.3}$$

$$g: X \times U \times R \to Y. \tag{2.4}$$

Usually, we assume that f, g are C^∞ maps, This hypothesis may, of course, be weakened in various directions.

We shall denote such a "system" by

$$\sigma = (U, X, Y, f, g). \tag{2.5}$$

An input-output pair is a curve $t \to (u(t), y(t))$ in $U \times Y$ such that there exists a curve $t \to x(t)$ in state space with

$$t \to (u(t),\ x(t),\ y(t))$$

a solution of (2.1)-(2.2). In the linear case, I have presented in Volume VIII an extensive discussion of these concepts, and hence there is no need to repeat that here. It suffices to say that the prime goal is to study the systems via their set of input-output pairs.

A good way to think of a system is that used in the engineering textbooks, namely via a "block diagram"

$$\xrightarrow{\quad u \quad} \boxed{\ x\ } \xrightarrow{\quad y \quad} \tag{2.6}$$

One thinks of a "signal" u coming into the "black box" labelled "x", acted on in some way, and then coming out as the signal y. In the case of systems modelled by the differential equations (2.1)-(2.2), the output y is obviously determined by the input curve $t \to u(t)$, <u>and</u> an initial state vector x_0. This is why the elements of X are called <u>state vectors</u>.

3. INTERCONNECTION OF SYSTEMS

Suppose $\sigma_1, \ldots, \sigma_n$ are systems of the type described in Section 2. Suppose σ_1 belongs to Σ_1, a collection of certain type of systems, σ_2 belongs to Σ_2, another type of system, and so forth. For example, Σ_1 might be all systems $(U_1, X_1, Y_1, f_1, g_1)$, with U_1, X_1, Y_1 fixed, and so forth. An <u>interconnection</u> is a mapping

$$\Sigma_1 \times \ldots \times \Sigma_n \to \Sigma_{n+1},$$

where Σ_{n+1} is another collection of systems.

The elementary theory of electric circuits and machines obviously presents many examples of interconnections. For example, think of a high-fidelity sound system, with all components linked together. Mechanical engineering presents many obvious examples, e.g. a car or a plane. In this section, I shall discuss what seems to be the two simplest examples, <u>series</u> and <u>parallel</u> interconnection.

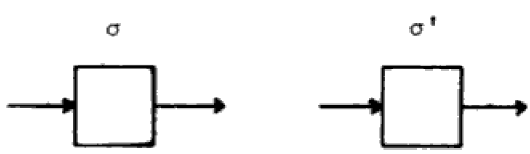
a) <u>Interconnection in series</u>

We shall discuss the "series" interconnection of two systems. The generalization to more should be obvious. Let:

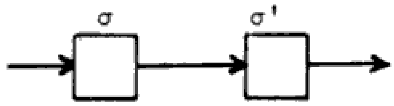
$$\sigma = (U, X, Y, f, g)$$

$$\sigma' = (U', X', Y', f', g')$$

be the two systems, denoted by their block diagrams:



To connect them "in series" is to hook them up, with the output of one equal to the input of the other:



(3.1)

This obviously requires that:

$$Y = U'$$

Let us write the system equations for

$$\sigma'',$$

the system resulting from hooking up σ and σ', as in (3.1):

Let us write down the system equations for σ and σ':

$$\sigma\colon \frac{dx}{dt} = f(x, u, t) \qquad y = g(x, u, t) \tag{3.2}$$

$$\sigma'\colon \frac{dx'}{dt} = f'(x', u', t) \qquad y = g'(x', u', t). \tag{3.3}$$

Hooking them up in series, as in (3.1), amounts to taking the output curve $t \to y(t)$ of σ, as determined by (3.2), and using it as the input to (3.3). This requires, of course, that

$$Y = U', \tag{3.4}$$

i.e. the output space of σ be the input space of σ'.

Setting:

$$u'(t) = y(t), \tag{3.5}$$

and substituting in (3.3), we have:

$$\begin{aligned} \frac{dx'}{dt} &= f'(x', g(x, u, t), t) \\ y' &= g'(x', g(x, u, t), t) \end{aligned} \tag{3.6}$$

Let us add to this first equation f (3.2):

$$\frac{dx}{dt} = f(x, u, t). \tag{3.7}$$

We see that (3.6) and (3.7) are the system equations for a system with input space U, output space Y', and state space $X \times X'$, i.e. u is the "input vector," (x, x') the state vector, y' the output vector. We can sum this up as follows:

Definition. Let σ and σ' be two systems, with input-state-output spaces (U, X, Y), (U', X', Y'). Then, the series-interconnection is a system σ', with input-state-output spaces (U, $X \times X'$, Y'), whose system equations are given by (3.6)-(3.7).

Of course, in this case we can form the system equations very readily by intuition. It would be desirable to have a general notion of "interconnection" or "interaction." We shall see later on how this can be done by regarding a "system" as an exterior differential system. We shall see that relation (3.5) may be regarded as a "constraint." In this form, there will be a close relation to the notion of "interaction," as considered in mechanics, thermodynamics, etc.

b). Interconnection in parallel

Again consider two systems

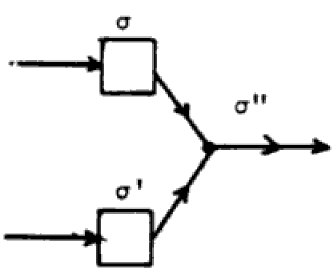
$$\begin{aligned} \sigma &= (U, X, Y) \\ \sigma' &= (U', X', Y'). \end{aligned} \tag{3.8}$$

(In the notation used here, a system is denoted explicitly by its spaces, with the system equations implicit). In a parallel interconnection, the outputs are added up. This requires, of course, that:

$$Y = Y'. \tag{3.9}$$

Remark: A more general possibility will be described below.

Here is the diagram:



(3.10)

Let us write this down more explicitly. Suppose that the system equations for σ and σ' are given by (3.2) and (3.3). Then, the output of σ'' is the curve

$$t \to y(t) + y'(t) \equiv y''(t).$$

The input is $t \to (u(t), u'(t))$, a curve in $U \times U'$. The state is $(x, x') \in X \times X'$. Thus,

$$\sigma'' = (U \times U', X \times X', Y). \tag{3.11}$$

The system equations of σ'' are:

$$\frac{d}{dt}\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} f(x, u, t) \\ g(x', u', t) \end{pmatrix} \tag{3.12}$$

$$y'' = f(x, u, t) + g(x', u', t). \tag{3.13}$$

<u>Remark</u>: Here is a common pattern to examples a) and b). The <u>Cartesian product</u> $\sigma \times \sigma'$ of two systems can be constructed as follows:

$$\sigma \times \sigma' = (U \times U', X \times X', Y \times Y').$$

The system equations of $\sigma \times \sigma'$ are just those of σ and σ' separately. Then, examples a) and b) are obtained by putting "constraints" on $\sigma \times \sigma'$.

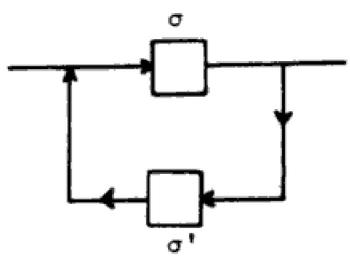
We now turn to another important example of interconnection, namely "feedback."

4. FEEDBACK INTERCONNECTION OF SYSTEMS

Suppose

$$\begin{aligned} \sigma &= (U, X, Y) \\ \sigma' &= (U', X', Y') \end{aligned} \tag{4.1}$$

are two systems, with indicated input-state-output spaces. The system σ'' resulting from using σ' as feedback into σ is that defined by the following diagram:



(4.2)

Here is what is meant by this diagram: First write down separately the system equations of σ and σ':

$$\begin{aligned} \frac{dx}{dt} &= f(x, u, t) \\ y &= g(x, u, t) \\ \frac{dx'}{dt} &= f'(x', u', t') \\ y' &= g'(x', u', t') \end{aligned} \tag{4.3}$$

We suppose that:

$$Y = U', \tag{4.4}$$

i.e. the output space of σ equals the input space of σ'. Also, that:

$$Y' = U, \tag{4.5}$$

i.e. that the output space of σ' equals the input space of σ.

Let σ'' be the system denoted by diagram (4.2). σ'' will have input-state-output spaces

$$(U,\ X \times X',\ Y).$$

To calculate the system equations of σ'', let us follow through the intuition suggested by (4.2).

A signal $t \to u(t)$ comes in. The output $y'(t)$ of σ' is subtracted, to form the signal

$$t \to u(t) - y'(t).$$

It goes through the block box of σ, and comes out as $t \to y(t)$, determined by the following equations:

$$\begin{aligned} y(t) &= g(x,\ u(t) - y'(t),\ t) \\ \frac{dx}{dt} &= f(x,\ u(t) - y'(t),\ t). \end{aligned} \tag{4.6}$$

It then goes through the block box of σ', coming out $y'(t)$, given as follows:

$$\begin{aligned} y'(t) &= g'(x',\ y(t),\ t) \\ \frac{dx'}{dt} &= f'(x',\ y(t),\ t). \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), we have:

$$\frac{dx'}{dt} = f'(x', g(x, u-y', t), t)$$

$$\frac{dx}{dt} = f(x, u-y', t) \tag{4.8}$$

$$y' = g'(x', g(x, u-y', t), t).$$

The system equations of σ'' are obtained by solving equations (4.7) for y, and substituting into (4.8). Since all this is confusing in the general case, I shall now do it all over for linear systems.

5. FEEDBACK FOR LINEAR SYSTEMS

Suppose the systems σ, σ' are linear, and stationary, as described in Volume VIII. The system equations for σ and σ' then take the following form:

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{5.1}$$

$$\begin{aligned} \frac{dx'}{dt} &= A'x' + B'u' \\ y' &= C'x' + D'u' \end{aligned} \tag{5.2}$$

The input-state-output spaces are real linear vector spaces. Here are the domains and ranges of the linear maps appearing on the right hand side of (5.1):

$$\begin{aligned} A&: X \to X \\ B&: U \to X \\ C&: X \to Y \\ D&: U \to Y. \end{aligned} \tag{5.3}$$

This situation can be summed up by the following diagram of vector spaces and linear maps. (Note that it is <u>not</u> a commutative diagram!)

$$\begin{tikzcd} U \arrow[r, "D"] \arrow[d, "B"'] & Y \\ X \arrow[r, "A"'] & X \arrow[u, "C"'] \end{tikzcd} \tag{5.4}$$

The system σ is often denoted as follows:

$$\sigma = (A, B, C, D). \tag{5.5}$$

<u>Remark</u>: For linear systems treated from an algebraic point of view, see Volume VIII, titled "Linear systems and introductory algebraic geometry." For a differential algebraic point of view, see the book by Kalman, Arbib and Falb [1]. Here are some standard mathematical objects.

<u>Definition</u>. Let $\sigma = (A, B, C, D)$ be a linear system. Let t be a real variable, and let s be a complex variable. The function

$$t \to Ce^{tA}B \equiv IR(\sigma) \tag{5.6}$$

is called the <u>impulse response</u> of the system. The function

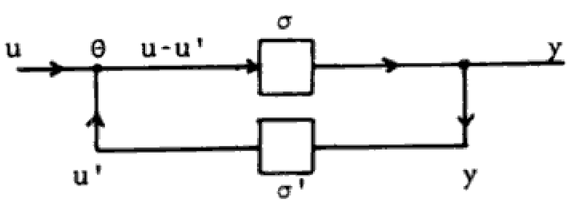
$$s \to C(s-A)^{-1}B \equiv FR(\sigma) \tag{5.7}$$

is called the <u>frequency response</u> of the system.

It is well-known that either of these two functions, plus D, serves to characterize the input-output properties of the system. $FR(\sigma)$ is also the <u>Laplace transform</u> of $IR(\sigma)$.

<u>Exercise</u>. Let σ'' be the system resulting from connecting σ and σ' in "series" r "parallel," as explained in Section 3. Calculate the impulse and frequency responses of σ''.

Suppose σ'' is the system obtained by using σ' as feedback into σ, as explained in Section 4. Here is the relevant block diagram.



(5.8)

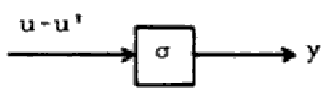
The object is to find the relation between y and u, and <u>define</u> σ'' as the system which provides this relation as its input-output relation.

Now, the symbol θ (read "subtractor") means that the signals u, u' coming into the vertex are to be subtracted. Of course, in order that this diagram makes sense, we must have:

$$\begin{aligned} Y' &= U \\ U' &= Y, \end{aligned} \tag{5.9}$$

i.e. the input space of σ is the output space of σ', and the input space of σ' is the output space of σ.

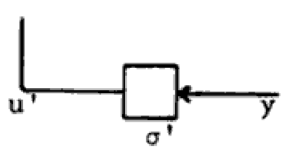
First, the relation



means that there is a curve $t \to x(t)$ in X, the state space of σ, such that:

$$\begin{aligned} \frac{dx}{dt} &= Ax + B(u-u') \\ y &= Cx + D(u-u'). \end{aligned} \tag{5.10}$$

Second, the relation



means that there is a curve $t \to x'(t)$ in X', the state space of σ', such that:

$$\frac{dx'}{dt} = A'x' + B'y' \tag{5.11}$$

$$u' = C'x' + D'y. \tag{5.12}$$

Let us insert (5.12) in (5.10):

$$\frac{dx}{dt} = Ax + B(u - C'x' - D'y) \tag{5.13}$$

$$y = Cx + D(u - C'x' - D'y). \tag{5.14}$$

Notice that u' does not appear in these equations. (5.11), (5.13), (5.14) are the desired system equations for σ''. Let us write them in standard form. First solve for y from (5.14).

$$(1 + DD')y = Cx + Du - DC'x'. \tag{5.15}$$

Insert (5.15) into (5.13) and (5.11):

$$\frac{dx'}{dt} = Ax' + B'(1 + DD')^{-1}(Cx + Du - DC'x') \tag{5.16}$$

$$\begin{aligned}\frac{dx}{dt} = {} & Ax + Bu - BC'x' \\ & - BD'(1 + DD')^{-1}(Cx + Du - DC'x').\end{aligned} \tag{5.17}$$

Let us sum up as follows:

Theorem 5.1. The system σ'', obtained by applying σ' as feedback to σ, has $X \times X'$ as state space, U as input space, and Y as output space, and has the following set of system equations:

$$\frac{dx}{dt} = (A - BD'(1 + DD')^{-1}C)x$$
$$- (BC' - BD'(1 + DD')^{-1}DC')x' \tag{5.18}$$
$$+ (B - BD'(1 + DD')^{-1}D)u$$

$$\frac{dx'}{dt} = B'(1 + DD')^{-1}Cx$$
$$+ (A - B'(1 + DD')^{-1}DC')x' \tag{5.19}$$
$$+ B'(1 + DD')^{-1}Du$$

$$y = (1 + DD')^{-1}Cx$$
$$- (1 + DD')^{-1}D'C'x' + Du. \tag{5.20}$$

Remark: The system equations (5.18)-(5.20) of the feedback system σ'' are forbiddingly complicated. Here is a simpler special case.

Assume that σ', the system used as feedback, has no state variables, i.e.

$$x' = 0. \tag{5.21}$$

Then,

$$B' = A' = C' = 0. \tag{5.22}$$

Thus, "x" is also the state vector for σ''. Equations (5.18)-(5.20) simplify to the following system equations for σ'':

$$\frac{dx}{dt} = (A - BD'(1 + DD')^{-1}C)x$$
$$+ (B - BD'(1 + DD')^{-1}D)u \tag{5.23}$$

$$y = (1 + DD')^{-1}Cx + Du. \tag{5.24}$$

A further simplication presents itself if:

$$D = 0. \tag{5.25}$$

Then, the equations (5.21)-(5.22) take the following form:

$$\frac{dx}{dt} = (A - BD'C)x + Bu \tag{5.26}$$

$$y = Cx. \tag{5.27}$$

In this case, the system

$$\sigma'' = (A - BD'C, B, C)$$

is related to the original system $\sigma = (A, B, C)$ by a very simple linear algebraic transformation, namely replacing:

$$A \to A - BD'C \tag{5.28}$$

These two systems are said to be <u>feedback equivalent</u>. Kalman and Rosenbrock (and many others) have extensively studied the algebraic properties of this sort of equivalence. To the best of my knowledge, there is not much known in <u>general</u> about analogous questions for the general equations (5.18)-(5.20).

6. INPUT-OUTPUT SYSTEMS AS EXTERIOR DIFFERENTIAL SYSTEMS

Let X, Y, U again be real, finite dimensional vector spaces, and consider an input-output system in the usual way:

$$\begin{aligned} \frac{dx}{dt} &= f(x, u, t) \\ y &= g(x, u, t). \end{aligned} \tag{6.1}$$

Now, mathematically, (6.1) is just a system of ordinary differential equations. We know that Cartan's theory of exterior differential systems is general enough to describe _any_ system of differential equations, ordinary or partial. It is then _possible_ to rewrite (6.1) as an exterior system - the reader may ask, however, why?

I can give two reasons. First, as we have seen in connection with classical mechanics and thermodynamics, the "functorial" transformation properties of differential forms under differentiable mappings makes them ideal differential-geometric objects to describe "interaction" and "constraints." (This is, in fact, one source of the superiority of differential geometry of Cartan's methods over those of classical tensor analysis - the latter is really only covariant under _diffeomorphisms_). We shall see that the sort of "interconnections" of systems described in previous sections may be described in a similiar way. Second, putting a system of differential equations into exterior differential form sets it up to describe various "structure," "isomorphism" and "symmetry" questions. Many examples of such questions are scattered throughout Cartan's work - most prominently in his magnificent but largely untapped theory of "infinite dimensional Lie groups" - and I plan to show (ultimately) that many of the most important and interesting questions of systems theory can be described within this framework.

At the moment, I will only discuss how the input-output system (6.1) may be described as an exterior system.

Suppose:

$$\dim X = n, \ \dim U = m, \ \dim Y = p.$$

Introduce indices with the following ranges, and the corresponding summation convention on these indices:

$$1 \le i, j \le n$$
$$1 \le a, b \le m$$
$$1 \le \alpha, \beta \le p.$$

Thus, $x = (x_i)$, $u = (u_a)$, $y = (y_\alpha)$ denote points of X, U, Y.

Let T be the space of the time variable t. Set:

$$M = X \times U \times Y \times T.$$
$$= \text{space of variables } (x, u, y, t).$$

Let I be the differential ideal of differential forms on M generated by the following 0 and 1-forms:

$$\begin{aligned} &dx_i - f_i dt \\ &y_\alpha - g_\alpha . \end{aligned} \tag{6.2}$$

Definition. A curve

$$s \to \sigma(s)$$

in M (i.e. a map $R \to M$) is an integral curve of I if the following condition is satisfied:

$$\sigma^*(\theta) = 0 \tag{6.3}$$

for all $\theta \in I$.

σ is said to be transversal to a real-valued function t on M if:

$$\sigma^*(dt) \neq 0 \tag{6.4}$$

σ is said to be parameterized by t if:

$$\sigma^*(t) = t. \tag{6.5}$$

Remark: It is readily seen that a reparameterized integral curve of I is again an integral curve. Thus, if (6.4) is satisfied, σ can be reparameterized to satisfy (6.5).

Theorem 6.1. Let I be the differential ideal of forms on

$M = X \times U \times Y \times T$ generated by the zero-th and first degree forms given by (6.2). Then, there is a one-one correspondence between the solutions of the input-output system (6.1) and the integral curves of I which are parameterized by t.

Proof. A curve in M which is parameterized by t is one of the following forms:

$$t \to (x(t),\ u(t),\ y(t),\ t),$$

where $t \to x(t)$, $u(t)$, $y(t)$ are curves in X, U, Y. To say that such a curve is an integral curve of I is to say that:

$$d(x_i(t)) = f_i(x(t),\ u(t),\ t)dt,$$

i.e.

$$\frac{dx}{dt} = f(x(t),\ u(t),\ t),$$

and that:

$$\sigma^*(y_\alpha - g_\alpha), \text{ i.e. that}$$

$$y(t) = g(x(t),\ u(t),\ t).$$

This means, of course, that

$$t \to (x(t),\ u(t),\ y(t))$$

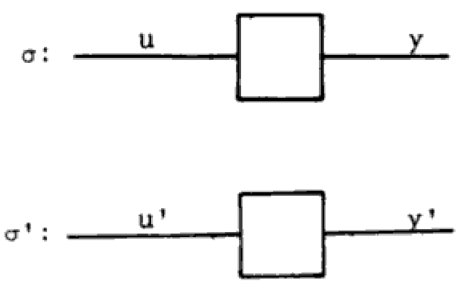
is a solution of the input-output system equations (6.1). The steps are obviously reversible, to prove that any solution of (6.1) leads in this way to an integral curve of

I which is parameterized by t.

Now, we shall explain this formulation of a "system" as an exterior system in order to formulate the "interconnection" notion in a very natural way.

7. INTERCONNECTION IN SERIES IN TERMS OF EXTERIOR SYSTEMS

Let us consider two input-output systems. For simplicity of notation, suppose they have the real numbers R as input and output, and state spaces.



(7.1)

Suppose the system equations are:

$$\begin{aligned} \frac{dx}{dt} &= f(x, u, t) \\ y &= g(x, u, t) \\ \frac{dx'}{dt} &= f'(x', u', t') \\ y' &= g'(x', u', t'). \end{aligned} \tag{7.2}$$

Set:

$$M = X \times U \times Y \times T$$

$$M' = X' \times U' \times Y' \times T'$$

We know how to convert these systems into exterior systems I and I' on M and M':

$$I = \{dx - fdt, y-g\}$$

$$I' = \{dx' - f'dt', y' - g'\}.$$

Notice that we introduce different "time" parameters for T and T'. Now, set:

$$\begin{aligned} M_0 &= M \times M' \\ &= (X \times X') \times (U \times U') \times (Y \times Y') \times (T \times T'). \end{aligned} \tag{7.3}$$

$$\begin{aligned} I_0 &= \{dx - fdt, dx' - f'dt', y-g, y'-g'\} \\ &\quad \text{ideal of forms generated by I and I'.} \end{aligned} \tag{7.4}$$

Let:

$$\begin{aligned} M'' &= \text{subset of } M_0 \text{ on which} \\ &\quad t = t', y = u' \end{aligned} \tag{7.5}$$

$$I'' = I_0 \text{ restricted to } M''. \tag{7.6}$$

M" can then be identified with:

$$(X \times X') \times (U \times Y') \times T. \tag{7.7}$$

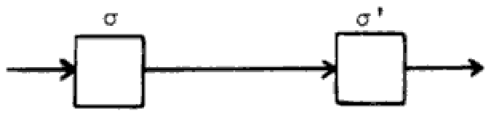
I" is generated by the following forms:

$$dx - f(x, u, t)dt$$

$$dx' = f'(x', y', t)dt$$

$$y' - g'(x', g(x, u, t), t)$$

We see that I" is the ideal of forms on M" associated with the <u>series interconnection</u> of σ and σ', as symbolized by the following diagram:



(7.8)

<u>In general</u>, here is the procedure to be followed to introduce "interactions" between input output systems σ and σ'. Let (M, I) and (M', I') be the exterior systems which are equivalent to the systems σ and σ'. Set:

$$M_0 = M \times M'$$

$$I_0 = I + I'$$

Introduce

$$M'' \subset M \times M' \tag{7.9}$$

as a submanifold, on which:

$$t = t',$$

$$I'' = I_0 \text{ restricted to } M''.$$

Of course, exactly <u>how</u> one chooses the submanifold M" determines what sort of "interconnection" on defines.

Exercise. Write down the details for case of interconnections of parallel and feedback interconnections, as described in previous sections.

Remark: Notice the similiarity in all of this to how "interactions" may be defined for thermodynamics and classical mechanics, as described in previous chapters. In turn, there are similiarities between these ideas and the ideas of G. Kron.

Remark: These ideas can also be readily described in terms of categories and functors. Indeed, this is probably the ultimate "correct" way of formulation, if one avoids the wretched excesses of abstraction that seem to afflict this area of mathematics. (My attitude toward category theory is that it is a good language for description of certain mathematical and physical phenomena, just as set theory is a good language for mathematics, but not something that I would want to teach grade-schoolers!)

Roughly, the input-output spaces should be considered as a category. The exterior systems form another category. (What the "morphisms" are should be made precise in each case, but I will not go into that here.) Then, our process of assigning exterior systems to an input-output system is a "functor" from the category of input-output systems to the category of exterior systems. "Interconnections" can now be described as algebraic structures on each of these categories, which is preserved by the functor.

8. STABILIZING LINEAR INPUT-OUTPUT SYSTEMS BY MEANS OF FEEDBACK

Obviously, an important problem in the theory of "interconnections" is to describe the properties of systems one can construct by interconnecting two systems with known properties. In fact, this is probably the fundamental problem of engineering!

Amazingly, not too much is known about this, beyond a few simple things, which turn out to be very important. As an illustration, I will consider the problem covered by the title of this section.

Consider a linear, time-invariant input-output system, say of the form:

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{8.1}$$

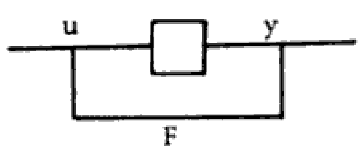
The system (8.1) is said to be <u>stable</u> if all eigenvalues of A (which maps $X \to X$, where X is the state vector space for the system) have negative real parts. Of course, the Routh-Hurwitz criterion (see Vol. II and Gantmacher [1]) gives conditions for this, but we shall not use it here. Instead, the complementary Liapounoff ideas will take over. Namely, we use the result that A is a stable linear map if and only if all solutions $t \to x(t)$ of the differential equations

$$\frac{dx}{dt} = Ax \tag{8.2}$$

go to zero as $t \to \infty$, i.e. that the system (8.2) is "asymptotically stable."

Exercise. Prove this equivalence between "stable linear map A" and "asymptotic stability" of (8.2).

Consider a simple "feedback" change of system (8.1), symbolized as follows:



As we have seen, in Section 5, the new system has the following system equations

$$\begin{aligned} \frac{dx}{dt} &= (A - BFC)x + Bu \\ y &= Cx. \end{aligned} \tag{8.3}$$

The new system (8.3) will be stable if all solutions of

$$\frac{dx}{dt} = (A - BFC)x \tag{8.4}$$

go to zero as $x \to \infty$.

Thus, the basic question is: Can one find a linear map

$$F: Y \to U \tag{8.5}$$

so that the new system, (8.3), constructed using F as "feedback" is stable. Here is the basic result.

Theorem 8.1. If the system (8.1) is completely controllable and observable, then F can be chosen so that the system (8.3) is stable.

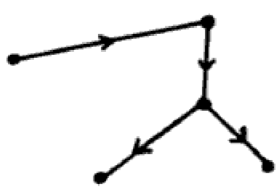
There are two proofs, one algebraic, the other analytic. The algebraic proof is given by Rosenbrock [1]. It seems difficult. There should be a more direct and simpler one, but I have not yet succeeded in finding it. The analytical proof proceeds by defining F via the solution of the matrix Riccati equation for the "regulator" problem associated with (8.1). See Brockett [1], Theorem 2 of Chapter 2.3. I will defer consideration of this proof until a later point in this work.

Exercise. Suppose the system (8.1) is scalar input-scalar output, i.e. U = R = Y, and is completely controllable and observable. Show directly that there is a "feedback" linear map $F: X \to U$ which changes the system to make it stable.

9. CLASSICAL AND GENERALIZED ELECTRIC CIRCUITS AS INTERCONNECTED EXTERIOR DIFFERENTIAL SYSTEMS

We have seen that "interconnections" and "interactions" of thermodynamic, mechanical and input-output systems may be defined in a reasonably unified way, when they are considered as exterior differential systems. To complete the "interconnection" story, I now show how electric circuits - which of course are the prototypical "system" - may also be considered in this way.

Refer to Volume III, Chapter I for the general setting of thermodynamics. Recall that an <u>oriented graph</u> is a triple (A, B, φ), where: A and B are sets. The elements of A are called <u>nodes</u>, the elements of B are called <u>branches</u>. The map $\varphi: B \to A \times A$ is called the <u>incidence map</u>. The graph is denoted geometrically in the usual way, with the branches as oriented line segments, the nodes as the end points



Let $b \in B$ be one branch. Associate with b a three dimensional manifold M(b). Denote the variables on M(b) by

$$i, v, t.$$

t is the _time_ variable, v is the _voltage_ variable, i is the _current_ variable. We shall define an ideal of differential form, denoted by

$$I(b),$$

on M(b). Physically, they will be equivalent to defining _constitutive relations_. The way the ideal I(b) is defined depends on whether the branch b is a _resistor_, _capacitor_, _inductor_, _current source_ or _voltage source_. In each case, I(b) is generated by a single 0 or 1-form θ:

Resistor

$$\theta = v - Ri \tag{9.1}$$

Capacitor

$$\theta = Cdv - idt \tag{9.2}$$

Inductor

$$\theta = Ldi - vdt \tag{9.3}$$

Current source

$$\theta = i - f(t) \tag{9.4}$$

Voltage source

$$\theta = v - f(t). \tag{9.5}$$

In (9.1)-(9.3), R, C, L are positive real constants, called the resistance, capacitance and inductance, respectively. In (9.3), (9.4) f(t) are assumed to be given functions of t, representing input currents or voltages.

Having defined the geometric structure which defines the physics at each branch, we now consider the interconnections of the branches. Set:

$$\begin{aligned} M' &= \text{Cartesian product of all spaces } M(b)\text{, as } b \text{ runs through the set of branches} \\ &\equiv \prod_{b \in B} M(b). \end{aligned} \tag{9.6}$$

The variables (v, i, t) are each M(b) then become functions on M', denoted by

$$v_b,\ i_b,\ t_b.$$

Set:

M = set of all points of M' at which all time variables t_b are equal.

Denote by "t" the common value of t_b on M'.

Let:

$$I = \text{exterior differential system on } M \text{ generated by the ideals } I(b). \tag{9.7}$$

So far, there is no notion of "interaction" between the different branches, beyond that expressed by equating all of the "time" variables. The Kirchoff laws provide such a notion.

Recall (e.g. from Volume II. See also Desoer and Kuh [1], Anderson and Vongpanitlerd, [1]. Seshu and Reed [1]) that there are two such laws, the Kirchoff current law and the Kirchoff voltage law, denoted by:

KCL and KVL.

In our way of looking at this, KCL and KVL will be submanifolds of the manifold M. To see how to define this submanifold, return to the graph

(B, A, φ).

Suppose (for simplicity) that B and A, the branch and node sets of the graph, are finite sets. Then, set:

$W(B)$ = vector space of formal linear combinations, with real coefficients, $W = \sum_{b \in B} x_b b$, of the branches

$W(A)$ = vector space of formal linear combinations of the nodes.

$W(B)^d$ = dual vector space of $W(B)$

$W(A)^d$ = dual vector space of $W(A)$.

Remark: As discussed in more detail in Volume II, $W(B)^d$ may be defined as the vector space of real valued functions on B, with W(B) then defined as its dual space.

The boundary map

$$\beta\colon W(B) \to W(A)$$

is now defined, using the incidence map $\varphi\colon B \to A \times A$. Thus, if

$$b \in B,$$

if

$$\varphi(b) = (a_1, a_2) \in A \times A,$$

then

$$\beta(b) = a_1 - a_2.$$

Return to circuit theory. We have defined the manifold M. The "currents" i_b, and "voltages" v_b are defined as real valued functions on M. For each point $p \in M$, we can define a element

$$w(p) \in W(B)$$
$$y(p) \in W(B)^d$$

as follows:

$$w(p) = \sum_{b \in B} i_b(p) \tag{9.8}$$

$$y(p)(b) = v_b(p). \tag{9.9}$$

w and y then define maps, with the following domains and ranges:

$$w: M \to W(B) \tag{9.10}$$

$$y: M \to W(B)^d . \tag{9.11}$$

<u>Definition</u>. KCL is the submanifold of points $p \in M$ such that:

$$\beta w(p) = 0 . \tag{9.12}$$

KVL is the submanifold of points $p \in M$ such that:

$$y(p) \in \beta^d (W(A)^d) \tag{9.13}$$

for all $p \in M$.

The exterior differential system I, restricted to the submanifold

$$(\mathrm{KCL}) \cap (\mathrm{KVL})$$

is then the basic mathematical object which determines the electric circuit.

Many generalizations of this formalism are possible. These are very important, because they lead one into the area of "mathematical modelling" for chemical and biological systems. The work by Katchalsky, Oster and Perelson is a pioneering effort here; much remains to be done. One immediate generalization is to suppose that the "currents" and "voltages" are "vector valued," rather than scalar valued. This presumably corresponds to what the electrical engineers call the "theory of n-ports." (See Anderson and Vongpanitlerd [1]). Here, one replaces W(B), the formal linear combination of the branch elements b with real coefficients by the space of formal linear combinations of branches b with elements in a vector space E(b). I can now readily explain what is involved here using the theory of vector bundles.

Let

$$\pi: E \to B$$

be a vector bundle over the space B, without fiber $\pi^{-1}(b) \equiv E(b)$ a real vector space. (B is a finite set say $B = (b_1, \ldots, b_m)$, so E is just a collection

$$E(b_1), \ldots, E(b_m)$$

of vector spaces. However, thinking in the "vector bundle" way might facilitate generalizations, e.g. to continuum theories) Let $\Gamma(E)$ denote the space of cross-section maps $:B \to E$. Since the fibers of E are vector spaces, elements of $\Gamma(E)$ may be added, i.e. $\Gamma(E)$ forms a real vector space. Now, set:

$$W(B, E) = \Gamma(E)^d \tag{9.14}$$

$\equiv$ dual space of the real space $\Gamma(E)$.

Exercise. In case $E = B \times R$, the product bundle of B with a one-dimensional fiber, show that

$$w(B, E) = W(B) \tag{9.15}$$

$\equiv$ the formal linear combination of the branches with real coefficients.

In case:

$$E = B \times U,$$

the product of B with a real vector space, show that:

$$w(B, E) = W(B) \otimes U \tag{9.15}$$

$\equiv$ the formal linear combinations of branches with vectors in U as coefficients.

We must now generalize the notion of "boundary operator" to fit into this generalization. (The fact that the vector spaces over branches may vary with the branch is a complication). I will not go into detail here. The correct formulation involves ideas (which have yet to be fully developed in this context) of the theory of "sheaves," which enables one, in general, to define homological ideas in the context where the coefficients are varying. I will simply postulate what is needed, namely a vector bundle

$$E' \to A$$

with the nodes as base space, and a "boundary" operator

$$\beta\colon W(B, E) \to W(A, E')$$

One now considers the subspace

$$\beta^{-1}(0) \times \beta^{d}(\Gamma(E')) \subset W(B, E) \times W(B, E)^{d},$$

and uses it to define the

$$(KCL) \cap (KVL)$$

subset of M in an analogous way. The key fact, from the abstract algebra point of view, is that:

$$\beta^{-1}(0) \subset W(B, E)$$

$$\beta^{d}(\Gamma(E') = W(A, E')^{d}) \subset W(B, E)^{d}$$

are orthogonal complements of each other with respect to the dual pairing between W(B, E) and its dual $W(B, E)^{d}$.

This generalization of the "electric circuit" ideas suggests a general differential-geometric way of formulating these ideas, which is independent of graph theory. I shall now describe such a theory.

10. AN ABSTRACT DIFFERENTIAL-GEOMETRIC VERSION OF ELECTRIC CIRCUIT THEORY

First, let us recall some elementary algebraic ideas concerning finite dimensional vector spaces and their duals. (See Volume II). Let V be a vector space, over the reals as field of scalars.

<u>Notational remark</u>. V should <u>not</u> be thought of as "voltages." In the circuit situation considered in Section 9, it may be thought of as W(B).

Let V^d denote the dual vector space to V. We can define a one-differential form θ on the <u>manifold</u>

$$V^d \times V,$$

called the <u>contact form</u>. To define it, suppose

$$(v_i),\ 1 \le i,\ j \le n = \dim V$$

is any basis of V. Let

$$(v_i{}^d)$$

be the dual basis of V^d, i.e.

$$v_i{}^d(v_j) = \delta_{ij}. \tag{10.1}$$

Let

$$x_i\colon V \to R$$

be the linear functions on V such that:

$$v = x_i(v)v_i, \tag{10.2}$$

for $v \in V$.

i.e. $x_i(v)$ is the i-th coordinate of v with respect to the basis $(v_1, \ldots, v_n)$ of V. Similiarly, let

$$y_i: V^d \to R$$

be the linear real valued functions such that:

$$v^d = y_i(v^d)v_i{}^d \tag{10.3}$$

for all $v^d \in V^d$.

Now, define the contact form θ on $V^d \times V$ as follows:

$$\theta = y_i dx_i. \tag{10.4}$$

<u>Exercise</u>. Show that θ is intrinsically defined on $V^d \times V$, in the sense that if (v_i') is another basis of V, giving use in a similiar way to functions

$$y_i', x_i',$$

then

$$y_i'dx_i' = y_i dx_i.$$

<u>Exercise</u>. Identify $V^d \times V$ with

$$T^d(V),$$

the <u>cotangent bundle</u> to the manifold V. With this identification show that θ is identical with the <u>contact one-form on the cotangent bundle to the manifold V</u>, as defined in VB, vol. II and GPS.

Remark: In classical mechanics, θ is more familiar when written as:

$$\theta = p_i dq_i,$$

where "q_i" are the configuration space coordinates, "p_i" are the momentum space coordinates.

Theorem 10.1. Let V' be a linear subspace of V, and let

$$(V')^{\perp} \subset V^d$$

be its orthogonal complement in the dual space, i.e.

$$(V')^{\perp} = \{v^d \in V^d \colon v^d(V') = 0\}. \tag{10.5}$$

Then, θ is zero on the subspace

$$V' \times (V')^{\perp}$$

of $V \times V^d$.

Proof. We can use the result of the above exercise, that is independent of basis chosen for V. Suppose the basis is chosen so that:

$$v_1, \ldots, v_m \text{ is a basis for } V'. \tag{10.6}$$

Then, also:

$$v_{m+1}{}^d, \ldots, v_n{}^d \text{ is a basis for } (V'). \tag{10.7}$$

Hence,

$$x_{m+1}, \ldots, x_n \text{ are zero on } V' \tag{10.8}$$

$$y_1, \ldots, y_m \text{ are zero on } (V')^{\perp} \tag{10.9}$$

(10.8) and (10.9) now imply that θ is zero on $V' \times (V')^{\perp}$.

Exercise. Show, conversely, that any linear subspace of dimension n of the direct sum vector space

$$V \oplus V^d,$$

on which the contact one-form θ is zero, is of the form

$$(V') \oplus (V')^{\perp}.$$

Definition. A linear subspace of $V \oplus V^d$, whose dimension is equal to the dimension of V, on which θ is zero, is called a Kirchoff subspace of $V \times V^d$.

Remark: Notice that, as a manifold, $V \times V^d$ is isomorphic to the vector space $V \oplus V^d$.

Now, we can give the basic concept.

Definition. A generalized circuit system is a quintuple (M, I, φ, V, K) consisting of:

a) A manifold M

b) An ideal I of differential forms on M, called the

<u>constitutive ideal</u>.

c) A map $\varphi\colon M \to V \times V^d$.

d) A <u>Kirchoff subspace</u> $K \subset V \times V^d$.

Given such an object, set:

$$M_K = \varphi^{-1}(K) \tag{10.10}$$

$$I_K = I \text{ restricted to } M_K. \tag{10.11}$$

A curve $\varphi\colon t \to M_K$ is then a <u>trajectory</u> of the system if it is an integral curve of I_K.

One can proceed to analyze the trajectory equations, generalizing what is done for electric circuits. (See Smale [1]). I plan to do this in a later volume. The main point I want to make here is to reemphasize the important role that differential forms can play in unifying conceptual problems in engineering and physics.

Chapter XII

THE THEORY OF NETWORK PORTS

1. INTRODUCTION

Electric circuits offer prototypes and examples of many sorts of abstract mathematical and physical structures. It is extremely useful and important to sort out such generalizations, since it seems that many situations - in biology, chemistry, economics, engineering and physics - can be modelled by means of these mathematical structures. In addition, it is important for the development of the overall program of this treatise to exhibit these structures in the clearest mathematical light, with emphasis on the characteristic algebraic and differential-geometric features.

In this chapter I shall investigate the relation between certain ideas in differential equation theory and the theory of network ports. My ultimate aim is to provide a useful differential-geometric setting for certain general ideas concerning "dissipative systems" developed by J. C. Willems [1] and certain ideas of Katchalsky, Oster and Perelson concerning generalization of network theory to biology and chemistry. Of course, clarifying the mathematical nature of differential-equations modelling diverse phenomena is a worthwhile goal in itself.

2. ORDINARY DIFFERENTIAL EQUATIONS SYSTEMS, AND NETWORK PORTS

Let M be a manifold. Let T denote the real numbers, parameterized by a "time" variable t,

$$-\infty < t < \infty .$$

Let T(M) denote the tangent bundle to M. If

$$t \to \sigma(t)$$

is a curve in M, then

$$t \to \sigma'(t) \in M_{\sigma(t)} \subset T(M)$$

denotes its tangent vector curve. The curve

$$t \to (\sigma'(t), t) \equiv \sigma^1(t) \in T(M) \times T$$

will be called the <u>first prolongation</u> of the curve σ.

<u>Remark</u>: In terms of Ehresmann's theory of jets, σ_1 may be identified with the <u>1-jet of the map</u> σ_1: $T \to M$. The jet formalism might be useful in further studies, but for the moment the more standard (in differential topology) "tangent bundle" formalism will suffice.

<u>Definition</u>. An (<u>ordinary</u>) <u>differential equation system for</u> <u>M</u> is a submanifold

$$S \text{ of } T(M) \times T.$$

A curve $t \to \sigma(t)$ in M is a <u>solution</u> of the differential equation system S if the first prolongation of σ lies in S, i.e. if:

$$(\sigma'(t), t) \in S \tag{2.1}$$

for all t.

Let us see what this means in the more familiar classical language. Suppose

$$(x_i),$$

$$1 \leq i, j \leq m = \dim M$$

are coordinates for M. Then, T(M) has coordinates, labelled

$$(x_i, \dot{x}_i),$$

defined as follows:

$$x_i(v) = dx_i(v) \tag{2.2}$$

for $v \in T(M)$.

<u>Remark</u>: This coordinate system might be called a <u>Newtonian</u> <u>coordinate system</u> for T(M). Physicall, the $\dot{x}_i$ represent the "velocities" of the particles whose configuration coordinates are the x_i.

Thus,

$$(x_i, \dot{x}_j, t)$$

denote coordinates of $T(M) \times T$. The submanifold $S \subset T(M) \times T$ may be defined <u>locally</u> by setting functions

$$f_1(x, \dot{x}, t), \ldots, f_s(x, \dot{x}, t)$$

of these variables equal to zero.

Suppose that $t \to \sigma(t)$ is a curve in M, and that

$$t \to x(t) = (x_i(t))$$

is the curve in R^m which represent the curve in the given coordinate system. Then,

$$t \to (x(t), \dot{x}(t) \equiv \frac{dx}{dt}, t) \tag{2.3}$$

are the coordinates of σ_1, the prolonged curve of σ, in the "Newtonian" coordinate system for $T(M) \times T$. Thus we see that:

> The curve in M whose coordinates are $t \to x(t)$ in R^m is a solution of the differential equation system S if and only if the curve (2.3) in R^{2m+1} satisfies the following system of ordinary differential equations:
>
> $$\begin{aligned} &f_1(x(t), \tfrac{dx}{dt}, t) = 0 \\ &\quad\vdots \\ &f_s(x(t), \tfrac{dx}{dt}, t) = 0. \end{aligned} \tag{2.4}$$

Thus, from a differential-geometric point of view, submanifolds of $T(M) \times T$ (or of the relevant "jet" space) are objects which play the role of "systems of ordinary differential equations", and are completely independent of choice of coordinates.

Let us turn now to describe what electrical engineers mean by an n-port. (n is an integer). In fact, I have not found a precise definition of this concept in the network theory literature. Roughly, it may be described as a set of n currents and voltages

$$(i_1, \ldots, i_n) = \underset{\sim}{i}$$

$$(v_1, \ldots, v_n) = \underset{\sim}{v},$$

part of a complicated electric circuit, for which the circuit equations always imply a fixed differential equation relationship. Thus, there should be functions

$$f_1(i, v, \dot{i}, \dot{v}, \ldots, t), \ldots, f_s(i, v, \dot{i}, \dot{v}, \ldots, t)$$

of the voltages, currents and their derivatives, such that the following differential equations hold:

$$\begin{aligned} &f_1(\underset{\sim}{i}(t), \underset{\sim}{v}(t), \frac{d\underset{\sim}{i}}{dt}, \frac{d\underset{\sim}{v}}{dt}, \ldots) = 0 \\ &\vdots \\ &f_s(\underset{\sim}{i}(t), \underset{\sim}{v}(t), \frac{d\underset{\sim}{i}}{dt}, \frac{d\underset{\sim}{v}}{dt}, \ldots) = 0 \end{aligned} \tag{2.5}$$

As a consequence of the overall circuit equations. Thus, for mathematical purposes, it will be sufficient to think of n-ports as systems (2.4) of ordinary differential equations, where the components

$$(x_1(t), \ldots, x_m(t))$$

of the vector x are identified with certain voltages or currents of a given circuit. Here are some examples.

Circuit elements as 1-ports

Electric circuits are based on graphs. To each branch of the graph, there is associated a voltage function $v(t)$ and a current function $i(t)$. Each such branch constitutes a circuit element, and also a 1-port, in the sense that there is a differential equation of the form (2.5) relating the voltage and current function. For the moment, we shall deal only with the simplest sort of circuit elements, i.e. linear, time-independent ones.

1) Resistors

$v = Ri$, or

$v - Ri = 0$.

This is the degenerate case of a differential equation systems, with the derivatives unconstrained.

2) <u>Capacitor</u>

The 1-port differential equation is:

$$i = C\frac{dv}{dt}$$

3) <u>Inductor</u>

$$v = L\frac{di}{dt}$$

4) <u>Voltage source</u>:

$$v = f(t)$$

5) <u>Current source</u>

$$i = f(t)$$

Electric circuit theory shows how to put together these "1-ports" to form general circuits. These rules should be generalized to n-ports. In turn, this will involve the idea of "interconnection" of n-ports, a topic to which we now turn.

3. INTERCONNECTION OF NETWORK PORTS

Let $P_1, \ldots, P_{m+1}$ be a collection of network ports of a certain type. An interconnection of $P_1, \ldots, P_m$, into P_{m+1}, is a mapping

$$P_1 \times \ldots \times P_m \to P_{m+1}.$$

We shall define various types of interconnections by means of the geometric concepts associated with exterior differential systems, just as we earlier defined interaction and interconnection of mechanical, thermodynamic and input-output systems. For simplicity, consider two such ports.

Suppose M and M' are manifolds.

Let

$$\pounds: T(M) \times T \to R^s$$

$$\pounds' : T(M') \times T \to R^{s'}$$

be mappings. Suppose that the first network port is defined by $(M, \pounds)$, i.e. the set of all $(v, t) \in T(M), \times T$ such that

$$\pounds(v, t) = 0 \tag{3.1}$$

defines port P_1. (3.1) is called the constitutive relation of the network port. Similiarly, suppose that

$$(M', \pounds')$$

defines the second port.

To define an "interconnection" between $(M, \pounds)$ and

$(M', \pounds')$, consider:

$$M \times M'.$$

Let

$$N \subset M \times M' \tag{3.2}$$

be a submanifold.

<u>Definition</u>. A network port $(M'', \pounds'')$ is said to be an <u>interconnection</u> of $(M, \pounds)$ and $(M', \pounds')$ if there is a map

$$\varphi: N \to M''$$

such that:

> φ defines a homomorphism of the differential equation system defined by $(\pounds, \pounds')$ on N into the differential equation system on M'' defined by $\pounds''$.

<u>Remark</u>: Recall what is meant by the term "homomorphism" between differential equations. (See Volume III), namely that φ maps a curve in N which is a solution of the ordinary differential equations determined by $\pounds$, $\pounds'$, into a solution which is a solution of $\pounds$. Explicitly, if

$$t \to \sigma(t),$$

$$t \to \sigma_1(t)$$

are curves in M, M' such that:

$$\mathfrak{L}(\sigma'(t), t) = 0$$

$$\underset{\sim}{\mathfrak{L}}'(\sigma_1'(t), t) = 0$$

$$(\sigma'(t), \sigma_1'(t)) \in N,$$

then

$$\mathfrak{L}''(\varphi(\sigma(t), \sigma_1(t))', t) = 0.$$

Alternately, we can consider the first port defined by a exterior system I of forms on T(M) × T, the second by an exterior system I' on T(M') × T. Consider the exterior system I_0 on N × T obtained by restricting the form on I and I'. Then, we require that

$$\varphi: N \to M'' \times T$$

is a homomorphism between the system I_0, I", in the sense that an integral curve of I_0 is mapped into an integral curve of I".

4. SERIES AND PARALLEL CONNECTION n-PORTS

Let (M, $\underset{\sim}{\mathfrak{L}}$) define a network port. Suppose:

$$\dim M = 2n. \qquad (4.1)$$

An n-port structure for (M, $\underset{\sim}{\mathfrak{L}}$) is defined by exhibiting M as the product

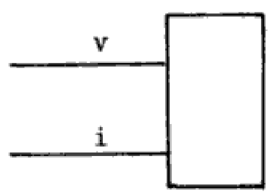
$$M = C \times V$$

of vector spaces C, V with:

$$\dim C = n = \dim V. \tag{4.2}$$

C is called the <u>current manifold</u>, V is called the <u>voltage manifold</u>.

A point of C is usually denoted by i, which is of course the usual electrical engineer's notation for <u>current</u>. Similiarly, a point of V is denoted by v, which is the usual notation for <u>voltage</u>. Here is the usual way to denote such an n-port:

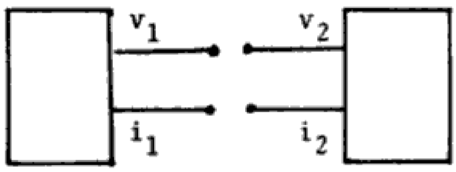


(4.3)

Of course, such a system can be converted into an input-output system by choosing one of the pair (i,v) as "inputs," the other as "output."

Let us now define the <u>series</u> and <u>interconnection</u> of such n-ports. Consider two such n-ports

$$M_1 = C_1 \times V_1, \; M_2 = C_2 \times V_2$$



Now, in series interconnection, the physical rules are equality of currents and addition of voltages. (This is a consequence of Kirchoff's laws, of course). This translates into the following rules for constructing the interconnection:

Set

$$N = \{((i_1, v_1), (i_2, v_2)) \in M_1 \times M_2 : i_1 = i_2\}$$

$$M = C \times V.$$

Map

$\varphi: N \to M$ as follows

$$\varphi((i, v_1), (i, v_2)) = (i, v_1 + v_2)$$

for $i \in C = C_1 = C_2$,

$v_1, v_2 \in V$.

To define the constitutive relations for the series interconnection of P_1, P_2, suppose, for example, that the constitutive relations for P_1, P_2 can be put into the following form:

$$\frac{dv_1}{dt} = g_1(i_1, \frac{di_1}{dt})$$

$$\frac{dv_2}{dt} = g_2(i_2, \frac{di_2}{dt}).$$

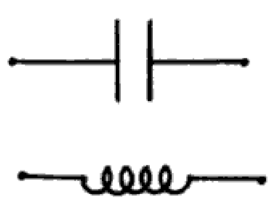
Then, the constitutive relation for the series interconnection are:

$$\frac{dv}{dt} = g_1(i, \frac{di}{dt}) + g_2(i, \frac{di}{dt}).$$

Exercise. Define the parallel interconnection of two n-ports in a "dual" way, i.e. by equality of voltages and addition of currents. Draw a diagram for this interconnection which correlates this rule with Kirchoff's laws.

Example. Series connection of a capacitor and inductor

Consider two 1-ports, corresponding to the circuit elements "capacitor" and "inductor."



(4.4)

Denote the current and voltage in the first circuit by

$$i_1, v_1$$

and by i_2, v_2 for the second. Denote by (i, v) the current-voltage in the series-connected 1-port. Here are the constitutive relations for the 1-ports (4.4):

$$i_1 = C \frac{dv_1}{dt} \qquad v_2 = L \frac{di_2}{dt} \tag{4.5}$$

C and L are positive constants, the capacitance and inductance.

Now, "series interconnection" means that, first, the equality of current constraint (i.e. KCL) is placed on the product space $M_1 \times M_2$, i.e. the space of variables $((i_1, v_1), (i_2, v_2))$. This leads to the space of variables

$$(i, v_1, v_2) \tag{4.6}$$

Now, KVL determines a map of the space of variables (4.6) into the space of variables (i, v), i.e.

$$(i, v_1, v_2) \to (i, v_1 + v_2 = v). \tag{4.7}$$

We must see which system of differential equations in the variables (4.7) has the property that the map (4.7) is a homomorphism from the system (4.5). To see this, note that

$$\begin{aligned} \frac{dv}{dt} &= \frac{dv_1}{dt} + \frac{dv_2}{dt} \\ &= \frac{i_1}{C} + L \frac{d^2 i_2}{dt^2} \\ &= \frac{i}{C} + L \frac{d^2 i}{dt^2} \end{aligned} \tag{4.8}$$

This differential equation is the constitutive relation for the series-connected 1-port.

Exercise. Compare to the constitutive relation for the parallel-connected 1-port.

As we have seen, the series and parallel connection of n-ports is, in a way, a special case of Kirchoff's laws, where one interprets "addition" of currents and voltages as vector addition. I will now describe a direct way to generalize this, to define one notion of interconnection for n-ports.

5. A GENERALIZED n-PORT CIRCUIT THEORY BASED ON KIRCHOFF'S LAWS

Let V be a real, finite dimensional vector space. Let V^d be its dual.

Remark: Using the appropriate functional analysis ideas, it would be possible to generalize to the case of infinite dimensional vector spaces. This generalization would be especially relevant to continuum mechanics.

Let (A, B, φ) be a directed graph. Recall (e.g. from Vol. III) that A, B are finite sets, with A the nodes, B the

<u>branches</u>, and

$$\varphi\colon B \to A \times A$$

the <u>incidence map</u>.

Let

$$F(B, V)$$

be the space of maps

$$\chi\colon B \to V.$$

An element $\chi \in F(B, V)$ is called a <u>voltage</u> for the circuit based on the graph (A, B, φ). For each branch b, denote by

$$v_b$$

the value of χ on b. This element of the vector space V is called the <u>voltage through the branch b</u>.

Now, let

$$\begin{aligned} I(B, V) &= F(B, V)^d \\ &= \text{dual space of the vector space } F(B, V). \end{aligned}$$

An element i of $I(B, V)$ is called a <u>current</u> through the graph.

<u>Theorem 4.1</u>. $I(B, V)$ may be identified with the space of formal linear combinations

$$\sum_{b \in B} v_b^d \, b \tag{5.1}$$

of the branches b, with coefficients in the <u>dual vector space V^d</u>.

<u>Proof</u>. Each element of the form (5.1) determines an element of the dual space of F(B, v). The value of the formal linear combination (5.1) on a $\chi \in F(B, V)$ is defined as:

$$\sum_{b \in B} v_b^d(\chi(b)). \tag{5.2}$$

<u>Exercise</u>. Show that this linear mapping from the space of formal linear combinations (5.1) into $F(B, V)^d$ is <u>one-one</u>.

Hence, to finish the proof of Theorem 4.1, one has only to prove the following.

<u>Exercise</u>. Show that F(B, V) and the space of formal linear combination (5.1) have the same dimension.

<u>Definition</u>. Identifying I(B, V) with the space of formal linear combinations, the element

$$v_b^d \in V^d$$

appearing in (5.1) for each branch b is called the <u>current</u>

through the branch b, and is denoted by

$$i_b.$$

Thus, a current for the graph (A, B, φ) is a formal linear combination

$$i = \sum_{b \in B} i_b b \tag{5.2}$$

Remark: It is especially important to keep in mind that "voltages" and "currents" take values in dual vector spaces. Of course, if

$$\dim V = n,$$

then a "voltage vector" and a "current vector" in a branch b are both n-tuples of real numbers, i.e. define "n-ports" in the sense that this term is used by electrical engineers. This is especially important for differential-geometric purposes. If V is a vector space, recall that the manifold

$$V \times V^d$$

has two natural and important structures, a contact and a symplectic one, defined by a 1-form θ and its exterior derivative $d\theta$. The contact structure on

$$F(B, V) \times I(B, V)$$

plays the key role in Tellegen's Theorem, and the symplectic structure on $V \times V^d$ plays the key role in defining the concept of reciprocal constitutive relation, both important in circuit theory.

We can now define the Kirchoff laws, using the boundary operator β of the graph (A, B, φ). If $b \in B$ satisfies:

$$\varphi(b) = (a_1, a_2),$$

with $a_1, a_2 \in A$, then define:

$$\beta(i_b b) = i_b(a_2 - a_1).$$

Extend this by linearity to define the boundary map

$$\beta: I(B, V) \to \text{formal linear combinations of elements of } A \text{ with coefficients in } V^d.$$

The currents in the kernel of this boundary map β are said to satisfy Kirchoff's current law. The linear subspace of such elements of $I(B, V)$ is denoted by

KCL.

An element of $F(B, V)$ is said to satisfy Kirchoff's voltage law if it is in the orthogonal complement of XCL with respect to the dual pairing of $I(B, V)$ and $F(B, V)$, defined by formula (3.2). KVL denotes the subspace of $v \in F(B, V)$ which satisfy Kirchoff's voltage law.

<u>Exercise</u>. Show that:

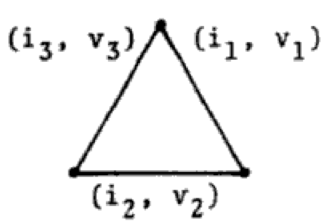
$$\text{KVL} = \beta^d(F(A, V)). \tag{5.3}$$

<u>Exercise</u>. Show that formula (5.3) implies the form of KVL which is described in the elementary (and now old-fashioned) text books:

The sum of the voltages around <u>closed</u> circuits are zero. (5.4)

For which sort of graphs does (5.4) holding for <u>all</u> closed circuits of the graph <u>imply</u> KVL? (This requires some deeper knowledge of graph theory, of course.)

<u>Example</u>: Suppose the graph is a triangle:



(5.5)

Then, KCV is:

$$i_1 = i_2 = i_3$$

KVL is:

$$v_1 + v_2 + v_3 = 0.$$

Now, by the very definition of KVL as the orthogonal complement of KCL, we have:

$$i(v) = 0 \tag{5.6}$$

for $i \in$ KCL, $v \in$ KVL.

Formula (5.6) (which is a tautology here) is called Tellegen's Theorem in the circuit theory books.

So far, we have been going exactly parallel to "ordinary" circuit theory, corresponding to interconnecting capacitors, resistors and inductors. The generalization, so far, is mathematically rather trivial, only replacing a one-dimensional vector space in the classical theory by an n-dimensional one. However, this generalization gives us much greater freedom to choose the differential equations defining the constitutive relation.

Definition. A set of constitutive relations for the circuit of n-ports is defined by giving, for each branch b, a set of ordinary differential equation for curves in the manifold $V \times V^d$.

Thus, a curve

$$t \to (i(t), v(t))$$

in I(B, V), F(B, V) is said to define a solution of the circuit theory equation, if the following conditions are satisfied:

$$\begin{aligned} \underset{\sim}{i}(t) &\in \text{KCL}, \\ \underset{\sim}{v}(t) &\in \text{KVL} \end{aligned} \tag{5.7}$$

for all

$$\underset{\sim}{f}_b(\underset{\sim}{i}_b(t),\ \underset{\sim}{v}_b(t),\ldots) = 0 \tag{5.8}$$

$$\text{for all } b \in B, \text{ all } t \in T,$$

where $\underset{\sim}{f}_b$ are the functions (vector-valued) which determine the differential equation - constitutive relation in each branch. (The terms ... on (5.8) denote derivatives of the currents and voltages in each branch.)

I will go more deeply into the differential-geometric meaning of the constitutive relations in the next sections. At this moment, let us only mention the simplest sort, the <u>resistive constitutive relations</u>. These are defined by the property that the functions $\underset{\sim}{f}_b$ do not involve the <u>derivatives</u> of $\underset{\sim}{i}$ and $\underset{\sim}{v}$. Thus, the relations

$$\underset{\sim}{f}_b = 0$$

determine a submanifold of

$$V \times V^d.$$

Conversely, any such submanifold defines a resistive constitutive relation.

As we have already mentioned, the natural <u>symplectic</u>

structure on $V \times V^d$ plays a key role in the classification of such constituitive relations. For example, the constitutive relation is said to be reciprocal (See Brayton [1]) if the following two conditions are satisfied:

The relations

$\mathcal{L}_b = 0$

determine a submanifold of dimension n of $V \times V^d$, where $\dim V = n$ (5.9)

The closed 2-form ω defining the symplectic structure on $V \times V^d$ is zero on the submanifold, i.e. the submanifold is a Lagrangian submanifold. (5.10)

Remark: If $(v_1, \ldots, v_n)$ are the usual "voltage" coordinates of V, $(i_1, \ldots, i_n)$ the dual "current" coordinates of V^d, then

$$\omega = di_1 \wedge dv_1 + \ldots + di_n \wedge dv_n.$$

The condition that $\omega = 0$ is then the usual form given in the literature. (See Brayton [1]. For example, if

$v \to g(v) = (g_i(v))$,

$1 \le i, j \le n$

is a map $V \to V^d$, and if the constitutive submanifold of $V \times V^d$ is the graph of this map, then the "reciprocal" condition is:

$$\frac{\partial g_i}{\partial v_j} = \frac{\partial g_j}{\partial v_i}. \tag{5.11}$$

Conditions like (5.11) appear at many places in physics, chemistry and engineering. (For example, in the <u>Onsager reciprocal relations</u> of "irreversible thermodynamics.") It seems that in each case they can be "understood" differential - geometrically by interpreting them as Lagrangian-submanifold conditions with respect to some naturally - defined "symplectic structure." Often making this structure explicit helps to understand the mathematics underlying the physical situation. Of course, this <u>mathematical</u> "isomorphism" between different physical situations is the heart of what I like to think of as "Interdisciplinary Mathematics."

6. THE LAGRANGE-RAYLEIGH EQUATIONS FOR N-PORTS

Now we shall study in greater detail the constitutive relations of a single n-port.

Let us begin with the simplest sort of ordinary differential equation which models physical situations:

$$a \frac{d^2q}{dt^2} + b \frac{dq}{dt} + cq = v(t). \tag{6.1}$$

a, b, c are constants. q is a real variable. (In mechanics, "q" is the "configuration space variable." In circuit theory, "q" is the "electric" charge). $t \to v(t)$ is the <u>forcing term</u>. In mechanics, it is usually "force" as it is known in physics since Newton's time, and in circuit theory it is usually "voltage." Introduce another variable $\dot{q}$, and set:

$$L = \frac{1}{2} a\dot{q}^2 - \frac{1}{2} cq^2 \tag{6.2}$$

$$R = \frac{1}{2} b\dot{q}^2. \tag{6.3}$$

L is the <u>Lagrangian</u>, R is the <u>Rayleighian</u> of the system. Consider the following <u>Lagrange-Rayleigh equations</u>:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial R}{\partial \dot{q}} = v(t). \tag{6.4}$$

$$\dot{q} = \frac{dq}{dt}. \tag{6.5}$$

Compute the left hand side of (6.4):

$$\frac{\partial L}{\partial \dot{q}} = a\dot{q}, \quad \frac{\partial L}{\partial q} = - cq.$$

$$\frac{\partial R}{\partial \dot{q}} = b\dot{q}.$$

Hence, (6.4) takes the form:

$$a \frac{d^2q}{dt^2} + b \frac{dq}{dt} + cq = v(t),$$

i.e. equation (6.1).

Now, from the circuit theory point of view, equation (6.1) is a typical <u>1-port constitutive relation</u>. Equation (6.4) is the <u>Lagrange-Rayleigh form</u> of this constitutive relation.

These relations are especially important in connection with the "energy" relations of the system

$$v(t) \frac{dq}{dt} = P(t) \tag{6.6}$$

is called the <u>power</u> of the system.

Define the <u>energy</u> as:

$$\begin{aligned} E(q, \dot{q}) &= \frac{\partial L}{\partial \dot{q}} \dot{q} - L \\ &= a\dot{q}^2 - \frac{1}{2} a\dot{q}^2 + \frac{1}{2} cq^2 \\ &= \frac{1}{2} a\dot{q}^2 + \frac{1}{2} cq^2. \end{aligned} \tag{6.7}$$

<u>Remark</u>: Notice that (6.7) reads (total) energy = kinetic + potential energy.

Suppose now that $t \to q(t)$ satisfies (6.1). Then,

$$\begin{aligned} \frac{d}{dt} E(q(t), \frac{dq}{dt}) &= a\dot{q}\,\ddot{q} + cq\dot{q} \\ &= \dot{q}(a\ddot{q} + c\dot{q}) \end{aligned}$$

$$= \dot{q}(f(t) - b\dot{q})$$

$$= P(t) - R(t).$$

Hence,

$$P(t) = \frac{d}{dt} E(q(t), \frac{dq}{dt}) + R(q(t), \frac{dq}{dt}). \qquad (6.8)$$

Relation (6.8) is an extremely basic relation from the viewpoint of general system theory. The fact that (6.8) holds for positive functions E and R is one way of stating the possitivity of the system. See J. C. Willems [1] for a fascinating general discussion of "passivity" and "dissipation" for general systems. In fact, a certain amount of my motivation in the work of this and the next few sections is the aim to generalize Willems' work to non-linear systems, and to better understand the differential-geometric nature and setting for the whole circle of ideas.

Now we can easily generalize. Let n be any integer, and choose indices and the summation convention as follows:

$$1 \le i, j, k \le n.$$

Let $q = (q_i)$, $i = (i_j)$, $v = (v_i)$ be real variables labelled by these indices, i.e. vectors in R^n. Let $L(q, \dot{q}, t)$, $R(q, \dot{q}, t)$ be functions of the indicated variables. Consider the following Lagrange-Rayleigh equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial R}{\partial \dot{q}} = v(t) \tag{6.9}$$

$$i = \frac{dq}{dt} \tag{6.10}$$

(Of course these are "vectorial" equations. For example:

$$\frac{\partial L}{\partial \dot{q}} \text{ stands for } \frac{\partial L}{\partial \dot{q}_i}$$

<u>Definition</u>. Consider an n-port constitutive relation involing the "current" and "voltage" variables i and v. This relation is said to be of <u>Lagrange-Rayleigh type</u>, with <u>Lagrangian</u> L and <u>Rayleighian</u> R, if relations (6.9)-(6.10) hold for any pair of curves $t \to (i(t), v(t))$ satisfying the n-port constitutive relations.

Let us now use the Lagrange-Rayleigh equations (6.9) to develop the energy-power relations. Set:

$$E(q, \dot{q}, t) = \frac{\partial L}{\partial \dot{q}} \dot{q} - L. \tag{6.11}$$

E is called the <u>total energy</u>.

$$P(\dot{q}, v) = \dot{q}_j v_j \tag{6.12}$$

$$\equiv qv.$$

P is called the <u>power</u> of the system.

Suppose now that

$$t \to (q(t), i(t), v(t))$$

is a curve in $R^n \times R^n \times R^n$ which satisfies the Lagrange-Rayleigh equations (6.9). Then,

$$\frac{d}{dt} E(q(t), \frac{dq}{dt}, t)$$

$$= \frac{d}{dt} (\frac{\partial L}{\partial \dot{q}})\dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q}$$

=, using (6.9),

$$(\frac{\partial L}{\partial q} - \frac{\partial R}{\partial \dot{q}} + v)\dot{q} + \frac{\partial L}{\partial \dot{q}} q$$

$$- (\frac{\partial L}{\partial \dot{q}} q + \frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial t})$$

$$= v\dot{q} - \frac{\partial R}{\partial \dot{q}} \dot{q} - \frac{\partial L}{\partial t}. \qquad (6.13)$$

To interpret this identity in conventional terms set:

$$P(t) = v(t)i(t) \qquad (6.14)$$

P(t) is called the power in the system. Then, (6.13) reads as follows:

$$P(t) = \frac{d}{dt} E(q(t), \frac{dq}{dt}) + \frac{\partial R}{\partial \dot{q}} \dot{q} + \frac{\partial L}{\partial t}. \qquad (6.15)$$

Definition. The system is said to be passive if:

$$E(q, \dot{q}, t) \geq 0 \qquad (6.16)$$

$$(\frac{\partial R}{\partial \dot{q}} \dot{q} + \frac{\partial L}{\partial t})(q, \dot{q}, t) \geq 0 \qquad (6.17)$$

for all $(q, \dot{q}, t)$.

<u>Remark</u>: This gives a definition of "passivity" for non-linear systems that generalizes the usual definition for linear ones. I do not know if this is a significant generalization.

<u>Research problem</u>. Discuss conditions (generalizing those presented by J. C. Willems [1], for example), where an "interconnection" of passive systems is passive.

<u>Examples</u>. <u>Capacitors and resistors</u>.

Suppose $n = 1$. First, suppose:

$$L = -\frac{1}{2} C^{-1} q^2, \quad R = 0.$$

Then relation (6.9) takes the following form:

$$v = \frac{q}{C},$$

or

$$i = C \frac{dv}{dt}. \tag{6.18}$$

(6.16) is the constitutive relation for a <u>capacitor</u>. C is the <u>capacitance</u>.

$$E = \frac{1}{2} C^{-1} q^2.$$

Hence, the system is <u>passive</u> if and only if

$$C > 0.$$

Now, suppose "dually":

$$L = 0, \ R = \frac{1}{2} R_o \dot{q}^2 \tag{6.19}$$

(R_o is a real number, of course.) Equations (6.9) are:

$$v = R_o \dot{q} = R_o i. \tag{6.20}$$

This is the resistor constitutive RELATION. R_o is the resistance

$$\frac{\partial R}{\partial \dot{q}} \dot{q} = R_o \dot{q}^2.$$

Hence, the system is passive if and only if

$$R_o > 0.$$

These well-known results have thus been derived from general principles, which are capable of vast generalization!

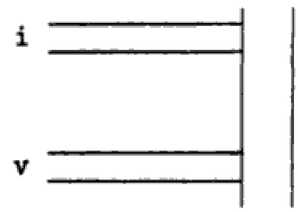
7. RECIPROCAL AND RAYLEIGHIAN RESISTIVE PORTS

Let

$$i = (i_1, \ldots, i_n) \in R^n$$

$$v = (v_1, \ldots, v_n) \in R^n$$

continue to be real n-vectors, representing the current and voltage of an n-port



Let $M = R^n \times R^n$, called the <u>current voltage manifold</u>. Recall that the "n-port" condition means that there is <u>some</u>: system of differential equations determining a family of curves

$$t \to i(t),\ v(t)$$

in M. Such an n-port is said to be <u>resistive</u> if the relation defining the n-port do not involve the derivatives of current and voltage, only their values. They are then of the form:

$$\mathcal{L}(i(t),\ v(t)) = 0, \tag{7.1}$$

where $\mathcal{L}$ is a map: $M \to R$.

Suppose now that such an n-port is <u>also</u> of Lagrange-Rayleigh type. Thus, there are "charge" variables

$$q = (q_1, \ldots, q_n),\ \dot{q} = (q_1, \ldots, q_n)$$

$$1 \le j,\ k \le n,$$

such that the relations (7.1) are equivalent to relations of the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial R}{\partial \dot{q}} = v \qquad (7.2)$$
$$\dot{q} = i.$$

Hence, the equations on the left hand side of (7.2) must be independent of q and $\ddot{q}$.

Exercise. Work out the conditions that L and R must satisfy.

Hint: If one considers the functions $\frac{\partial L}{\partial q_j}$, $\frac{\partial R}{\partial q_j}$ as Lagrangian and Rayleighian respectively, the Lagrange-Rayleigh equations are satisfied identically in q, $\dot{q}$. This forces $\frac{\partial L}{\partial q_j}$ and $\frac{\partial R}{\partial q_j}$ to be at most linear in $\dot{q}_j$.

One immediate way to satisfy (7.2) is to take L = 0, and R a function of $\dot{q} = i$ alone. We shall say that such a n-port is of Rayleigh type. The resistive constitutive relations between i and q then take the following form

$$v_j(i) = \frac{\partial R}{\partial i_j}. \qquad (7.3)$$

The integrability condition for such a relation are the following reciprocity conditions:

$$\frac{\partial v_j}{\partial i_k} = \frac{\partial v_i}{\partial i_k} \qquad (7.4)$$

A resistive n-port which is of Rayleigh type is then reciprocal, in the sense that it satisfies the conditions

(7.4). Of course, these conditions may be stated in the following far more elegant differential-geometric form.

> A <u>reciprocal resistive constitutive relation</u> between currents and voltages of an n-port is determined by an n-dimensional submanifold of R^{2n} such that
>
> $$di_j \wedge dv_k = 0$$
>
> on the submanifold i.e. the submanifold is a Lagrangian submanifold with respect to the natural symplectic structure on R^{2n}.

<u>Remark</u>: Of course, the further condition that the constitutive relation between currents and voltages may be written in form (7.3) is that the submanifold be transversal to the variables

$$(i_1, \ldots, i_n)$$

of R^{2n}.

<u>Exercise</u>. Is <u>every</u> resistive constitutive relation which is of Lagrange-Rayleigh type reciprocal?

8. CAPACITIVE CONSTITUTIVE RELATIONS

Continue with

$$i = (i_1, \ldots, i_n)$$

$$v = (v_1, \ldots, v_n)$$

as n voltage and current variables. If $t \to i(t)$ is a curve in the current space, a curve $t \to q(t)$ is called a <u>charge vector</u> if

$$\frac{dq}{dt} = i.$$

Introduce a <u>charge vector</u>

$$q = (q_1, \ldots, q_n),$$

and call the R^n labelled by it the <u>charge space</u>.

<u>Definition</u>. An n-port constitutive relation between i and v is said to be of <u>capacitive type</u> if there is a submanifold of (charge × voltage) space, such that, for each curve $t \to (i(t), v(t))$ satisfying the relation, the curve

$$t \to (q(t) = \int i(t)dt, v)$$

in charge × voltage space lies on the submanifold.

Following the pattern described in the previous section for resistors, let us find the condition that such a constitutive relation is <u>also</u> of Lagrange-Rayleigh type. These relations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial R}{\partial \dot{q}_j} = v_j \tag{8.1}$$

$$\dot{q} = i. \tag{8.2}$$

The capacitive relations then imply that the left hand side of (8.1) is independent of q, $\dot{q}$, $\ddot{q}$. This forces the following conditions:

$$\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_k} = 0. \tag{8.3}$$

$$\frac{\partial^2 R}{\partial q_j \partial \dot{q}_k} = 0 \tag{8.4}$$

$$\frac{\partial^2 R}{\partial \dot{q}_j \partial \dot{q}_k} = 0 \tag{8.5}$$

(8.3) and (8.4) can be solved in the following form:

$$L = A_j(t)\dot{q}_j + A(q, t) \tag{8.6}$$

$$R = B_j(q, t)\dot{q}_j + B(q, t). \tag{8.7}$$

Consider now a special relation of this form, namely:

$$\begin{aligned} L &= L(q, t) \\ R &= 0. \end{aligned} \tag{8.8}$$

Such a constitutive relation is called a <u>capacitive relation of pure Lagrangian type</u>. The relation (8.1) between v and q

then takes the following form:

$$v_j = -\frac{\partial L}{\partial q_j}. \qquad (8.9).$$

Again, this forces a constitutive relation between v and q of reciprocal type, in the sense that the submanifold in (voltage × charge) space defining the constitutive relation is a Lagrangian submanifold with respect to the symplectic structure defined by the 2-form:

$$dv_j \wedge dq_j$$

This proves the following result:

Theorem 8.1. A capacitive, n-port constitutive relation between v and q is of pure Lagrangian type if and only if it is reciprocal.

This result is again the key link between the "geometry" and the physics.

9. THE INDUCTANCE CONSTITUTIVE RELATIONS

So far, the current i and the voltage v play asymetric role. In terms of mechanics, "v" plays the role of "force," while "i" plays the role of "velocity." In the circuit constitutive relations of "inductor" type, these roles are reversed. In turn, this is related to what the circuit theorists call "duality." Mathematically, here is one facet what is involved.

Let V be a real vector space, V^d its dual space. (It is of course a basic idea of circuit theory that "voltages" and "currents" take values in such dual vector spaces). Then, there is a natural symplectic structure on the manifold.

$$V \times V^d,$$

which has played the basic role in the definition of "reciprocity." Now,

$$(V^d)^d \text{ is isomorphic to } V.$$

Hence, there is an automorphism (which is readily seen to be a automorphism of the symplectic structure) between

$$V \times V^d$$

and $$(V^d) \times (V^d)^d.$$

This is the mathematical source of the "duality" underlying circuit theory. It is also the underlying algebraic reason for the various sorts of "duality" (Poincaré, Alexander, Lefschetz ...) in algebraic topology. In fact, there is probably a very general sort of "duality" which includes the circuit theory and topological types as special cases.

In terms of "currents" i, "voltages" v, with

$$\omega = di \wedge dv,$$

this duality then corresponds to the transformation

$$i \to v$$
$$v \to -i.$$

Let us apply an elementary form of this duality to describe an <u>inductance n-port</u>. Again, let

$$i = (i_1, \ldots, i_n)$$
$$v = (v_1, \ldots, v_n)$$

be n-port <u>current</u> and <u>voltage</u> variables. Let

$$\varphi = (\varphi_1, \ldots, \varphi_n)$$

be additional variables, called the <u>flux</u> variables. If

$$t \to v(t)$$

is a curve in "voltage" space, the curve $t \to \varphi(t)$ in "flux" space such that:

$$\frac{d\varphi}{dt} = v(t)$$

is called the <u>flux curve</u>. A submanifold

$$N \subset R^{2n}$$

of flux x current space is called an <u>n-port inductance constitutive relation</u>. Such a constitutive relation is said to be <u>reciprical</u> if it is a Lagrangian submanifold (i.e. integral submanifold of maximal dimension) with respect to the 2-form:

$$\omega = d\varphi_j \wedge dq_j.$$

Any such submanifold N defines an <u>n-port inductance</u> relation: A curve $t \to (i(t), v(t))$ in current x voltage space satisfies the <u>n-port constitutive relation</u> if there is a curve $t \to (i(t), \varphi(t))$ in (current $\times$ flux) space such that:

$$(i(t), \varphi(t)) \in N$$

for all t.

$$v(t) = \frac{d\varphi}{dt}.$$

The n-port inductance is said to be <u>co-Lagrangian</u> if there is a function $L(\varphi)$ such that the relations

$$i = - \frac{\partial L}{\partial \varphi}$$

$$v = \frac{d\varphi}{dt}$$

determine the n-port relation between current and voltage.

<u>Exercise</u>. Show that reciprocity is again the condition that the inductance constitutive relation be co-Lagrangian.

10. THE TANGENT BUNDLE OF A SYMPLECTIC MANIFOLD

We can now abstract from these varied circuit-theory ideas some general differential-geometric ideas. Suppose

$$i = (i_1, \dots, i_n)$$

$$v = (v_1, \dots, v_n)$$

are n-port current and voltage variables. Introduce charge and flux variables:

$$q = (q_1, \dots, q_n)$$

$$\varphi = (\varphi_1, \dots, \varphi_n)$$

Differential-geometrically, the i and v are the "tangent bundle" variables for charge and flux, i.e.

$$i = \dot{q}, \; v = \dot{\varphi}.$$

The symplectic structure

$$\omega = d\varphi \wedge dq$$

plays a key role in circuit theory. (It is analogous to the role played by the form

$$dp \wedge dq$$

in classical mechanics, where "p" are the momentum variables, "q" are the position variables.)

This suggests that we start with a 2n-dimensional manifold M with a symplectic structure defined by a closed 2-form ω, and investigate how ω defines a symplectic structure on its tangent bundle

$$T(M).$$

Now, the "symplectic" condition means that ω is of <u>maximal rank</u>, i.e. for each point $p \in M$, ω defines a <u>non-degenerate</u>, skew-symmetric form

$$M_p \times M_p \to R.$$

Hence, there is a linear map

$$\alpha_p\colon M_p \to M_p{}^d$$

such that:

$$\alpha_p(v_1)(v_2) = \omega(v_1, v_2) \tag{10.1}$$

for $v_1, v_2 \in M_p$.

As p varies over M, we obtain a fiber-preserving map

$$\alpha\colon T(M) \to T^d(M).$$

It is readily seen that α is a diffeomorphism.

On $T^d(M)$, there is a ca-onical 1-form θ. (As explained in GPS and VB, II, θ defines the standard <u>contact structure</u> on T(M)). Set:

$$\theta' = \alpha^*(\theta) \tag{10.2}$$

$$\omega' = d\theta' = \alpha^*(d\theta) \tag{10.3}$$

θ' defines a <u>contact structure</u> on T(M), while ω' defines a <u>symplectic structure</u>. We shall see that these geometric objects play a key role in the geometry of constitutive relations.

Let us compute θ' and ω' in the notations that are customary in circuit theory. Suppose

$$\dim M = 2n.$$

Choose indices and the summation convention as follows:

$$1 \le j,\ k \le n.$$

Suppose $q = (q_j)$, $\varphi = (\varphi_j)$ are coordinates of M, with:

$$\omega = d\varphi_j \wedge dq_j. \tag{10.4}$$

Such a coordinate system is called a <u>canonical coordinate system</u> with respect to the symplectic structure.

Let

$$\dot{q}_j,\ \dot{\varphi}_j$$

denote the real valued functions

$$T(M) \to R$$

defined as follows:

$$\begin{aligned} \dot{q}_j(v) &= dq_j(v) \\ \dot{\varphi}_j(v) &= d\varphi_j(v) \end{aligned} \tag{10.5}$$

$$\text{for } v \in T(M).$$

Let

$$r_j,\ s_j\colon T^d(M) \to R$$

be defined as follows

$$r_j(v^d) = v^d\left(\frac{\partial}{\partial q_j}\right)$$
$$s_j(v^d) = v^d\left(\frac{\theta}{\partial \varphi_j}\right) \tag{10.6}$$
for $v^d \in T^d(M)$.

Consider (φ, q) as functions on $T^d(M)$ and $T(M)$, without any special notation, by pulling them back using the fiber space projection maps. Then,

$$\theta = r_j dq_j + s_j d\varphi_j. \tag{10.7}$$

To compute θ', we must compute $\alpha\colon T(M) \to T^d(M)$, using its defining formula (10.1)

$$\alpha\left(\frac{\partial}{\partial \varphi_j}\right)\left(\frac{\partial}{\partial \varphi_k}\right)$$
$$= \omega\left(\frac{\partial}{\partial \varphi_j}, \frac{\partial}{\partial \varphi_k}\right) = 0 = \alpha\left(\frac{\partial}{\partial q_j}\right)\left(\frac{\partial}{\partial q_k}\right)$$
$$\alpha\left(\frac{\partial}{\partial \varphi_j}\right)\left(\frac{\partial}{\partial q_k}\right) = \omega\left(\frac{\partial}{\partial \varphi_j}, \frac{\partial}{\partial q_k}\right)$$
$$=\text{, using (10.4), } \delta_{jk}$$
$$= -\alpha\left(\frac{\partial}{\partial q_j}\right)\left(\frac{\partial}{\partial \varphi_k}\right).$$

Hence,

$$\alpha\left(\frac{\partial}{\partial \varphi_j}\right) = dq_j$$
$$\alpha\left(\frac{\partial}{\partial q_j}\right) = -d\varphi_j \tag{10.8}$$

Now, using (10.8),

$$\alpha^*(s_j)(\frac{\partial}{\partial\varphi_k})$$

$$= s_j(dq_k) =, \text{ using (10.6)},$$

$$dq_k(\frac{\partial}{\partial\varphi_j}) = 0$$

$$\alpha^*(s_j)(\frac{\partial}{\partial q_j}) = - s_j(d\varphi_j)$$

$$= - d\varphi_k(\frac{\partial}{\partial\varphi_j}) = - \delta_{jk}.$$

Hence,

$$\alpha^*(s_j) = - \dot{q}_j. \tag{10.9}$$

Similiarly,

$$\alpha^*(r_j) = \dot{\varphi}_j. \tag{10.10}$$

Finally then,

$$\theta' \equiv \alpha^*(\theta) = \dot{\varphi}_j dq_j - \dot{q}_j d\varphi_j \tag{10.11}$$

$$\omega' \equiv d\theta' = d\dot{\varphi}_j \wedge dq_j - d\dot{q}_j \wedge d\varphi_j. \tag{10.12}$$

These are basic formulas for understanding the true differential-geometric meaning of the circuit theory constitutive relations! For as usual, let us write:

$$\dot{\varphi}_j = v_j \equiv \text{voltage variable} \tag{10.13}$$

$$\dot{q}_j = i_j \equiv \text{current variable.} \tag{10.14}$$

Thus, we see that, in the expression for ω', q is paired with v, i.e. "voltage" as dual to "charge." Similiarly, "current" is dual to "flux." We have seen already that a reciprocal capacitive constitutive relation corresponds to a Lagrangian submanifold of (charge $\times$ voltage) space, and that an inductance reciprocal constitutive relation is a Lagrangian submanifold of (flux $\times$ current) space. (Recall that q is the "charge," and φ the "flux," variable). Formula (10.12) now makes it clear why these are the geometrically correct pairings!

In fact, there should, geometrically, be a notion of combined capacitive-inductance constitutive relation which is a Lagrangian submanifold of ω! Such a theory would exploit to the full the "duality" between current-voltage and charge-flux. I will continue with the development of such a unifying theory.

11. THE DUALITY BETWEEN CHARGE AND FLUX PARAMETERIZED n-PORTS

As we have just seen, the "charges" and "fluxes" play a dual role in terms of the underlying differential geometry. The question arises whether this mathematical interpretation is natural from the point of view of physics. We shall now investigate this point.

Start from the beginning.

$$i = (i_1, \ldots, i_n)$$

$$v = (v_1, \ldots, v_n)$$

are two sets of n-vectors, called <u>currents</u> and <u>voltages</u>. Generally, an <u>n-port</u> is a set of different - integral equations determining a set of curves $t \to (i(t), v(t))$ in $R^n \times R^n$. Such a curve will be called an <u>n-port curve</u>.

There are two distinct mathematical ways to describe such n-ports that seem to be useful for physical purposes. In the first, introduce additional variables

$$q = (q_1, \ldots, q_n),$$

called <u>charges</u>, and define the n-port curves as the set of solutions of a system of ordinary differential equations

$$\begin{aligned} & f\left(q, \frac{dq}{dt}, \frac{d^2q}{dt^2}, \ldots, v, t\right) = 0 \\ & i = \frac{dq}{dt} \end{aligned} \tag{11.1}$$

(Of course, "f" stands for a set of functions, so that (11.1) is really a "system" and not a single differential equation.) The second way is to introduce "dual" <u>flux</u> variables

$$\varphi = (\varphi_1, \ldots, \varphi_n),$$

and define the n-port curves by means of a system of differential equations of the following form:

$$\begin{aligned} &q(\varphi, \frac{d\varphi}{dt}, \frac{d^2\varphi}{dt^2}, \ldots, i, t) = 0 \\ &v = \frac{d\varphi}{dt}. \end{aligned} \tag{11.2}$$

Now the same physical system, considered in the "n-port" way as a set of curves in current × voltage space, might be described in either the form (11.1) or the form (11.2). The first is called the charged-parameterized way, the second the flux-parameterized way. Our goal is to find the conditions that a given n-port admit such "dual" descriptions.

Remark. There is also the interesting additional possibility of the given n-part being parameterized partially by charges, partially by fluxes.

Let us work out the example where the charge parameterization is via the Lagrange-Rayleigh method. This means that there are functions

$$L(q, \dot{q}, t), \ R(q, \dot{q}, t)$$

such that the equations (11.1) are of the following form:

$$\begin{aligned} &v_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial R}{\partial \dot{q}_j} \\ &i_j = \frac{dq_j}{dt}, \end{aligned} \tag{11.3}$$

where the indices j, k,... are chosen to range from 1 to n.

Here is one way of thinking about this problem in terms of differential form system. Let M be the space of variables

$$(q, \varphi, \dot{q}, \dot{\varphi}, t).$$

<u>Remark</u>. As we have seen in Section 10, M has a contact structure which is naturally defined in terms of the symplectic structure

$$d\varphi \wedge dq,$$

so that what we say has a natural differential-geometric interpretation. Let I be the ideal of differential forms on M generated by the following 1-forms:

$$\begin{aligned} &dp_j + \left(\frac{\partial R}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} - \dot{\varphi}_j\right)dt \\ &dq_j - \dot{q}_j\ dt \\ &d\varphi_j - \dot{\varphi}_j\ dt \end{aligned} \tag{11.4}$$

where p_j is the function on M defined as follows:

$$p_j = \frac{\partial L}{\partial \dot{q}_j}. \tag{11.5}$$

Let us similiarly suppose that the n-port is flux-parameterized in the Lagrange-Rayleigh way. Recall that this means that there are functions

$$
\begin{aligned}
i_j &= \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\varphi}_j}\right) - \frac{\partial L'}{\partial \varphi_j} + \frac{\partial R'}{\partial \dot{\varphi}_j} \\
v_j &= \frac{d\varphi_j}{dt}
\end{aligned}
\qquad (11.7)
$$

Let I' be the ideal of differential forms on M generated by the following differential forms:

$$
\begin{aligned}
& dp_j' + \left(\frac{\partial R'}{\partial \dot{\varphi}_j} - \frac{\partial L'}{\partial \varphi_j} - \dot{q}_j\right)dt \\
& dq_j - \dot{q}_j dt \\
& d\varphi_j - \dot{\varphi}_j dt \\
& p_j' = \frac{\partial L'}{\partial \dot{\varphi}_j}
\end{aligned}
\qquad (11.8)
$$

Thus, we see that we have proved the following result:

<u>Theorem 11.1</u>. The given n-port is charge-parameterized by (L, R) and flux-parameterized by (L', R') if and only if I and I' have the same set of t-parameterized integral curves.

<u>Exercise</u>. Work out the implied conditions in case L, R, L', R' are time-independent and at most quadratic in the other variables. (This is, of course, the condition that the n-port differential equations are <u>linear</u>, which is the typical situation in circuit theory.)

I shall defer to a later point in the treatise any attempt to work out the conditions in the general non-linear case. Instead, I shall now present an elegant differential-geometric way to realize these relations, using the theory of the Cartan forms.

12. n-PORT RELATIONS IN TERMS OF CARTAN FORMS

Continue with the notations of Section 11. Let

$$L(q, \dot{q}, t), R(q, \dot{q}, t)$$

be Lagrangian and Rayleighian function, which define a charge-parameterized n-part.

Let M be the space of variables

$$(q, \dot{q}, v, t).$$

Set:

$$\theta(L) = Ldt + \frac{\partial L}{\partial \dot{q}_j}(dq_j - \dot{q}_j dt) \tag{12.1}$$

$\theta(L)$ is called the Cartan-form associated with the Lagrangian L. As I have already shown in LAQM, VB, and GPS, it plays a basic role in the study of the geometric properties of the calculus of variations. (A further important feature is that the methods may be generalized to multiple integral variational problems, which will provide generalizations to field-theoretical problems.)

Let

$$t \to (q(t), \dot{q}(t) = \frac{dq}{dt}, t, v(t)) = \sigma(t)$$

be a curve in these variables. Let $t \to \sigma'(t)$ be its tangent vector curve. Set:

$$\theta_j = dq_j - \dot{q}_j dt \tag{12.2}$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{12.3}$$

$$\eta_j = dp_j - \frac{\partial L}{\partial q_j} dt. \tag{12.4}$$

<u>Exercise</u>. Show that

$$d\theta(L) = \eta_j \wedge \theta_j. \tag{12.5}$$

Using (12.5), and the fact that:

$$\sigma'(t) \lrcorner \theta_j = 0, \tag{12.6}$$

we have

$$\sigma'(t) \lrcorner d\theta(L) = \eta_j(\sigma'(t))\theta_j \tag{12.7}$$

(12.7) is a Basic Formula. We have made no assumptions. Let us suppose that the curve $t \to q(t)$ satisfies the Lagrange-Rayleigh n-port equations, i.e. that:

$$v_j(t) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial R}{\partial \dot{q}_j}.$$

We see that:

$$\eta_j(\sigma'(t)) = (v_j(t) + \frac{\partial R}{\partial \dot{q}_j}.$$

Hence,

$$\sigma'(t) \mathbin{\lrcorner} d\theta(L) = v_j\theta_j + \frac{\partial R}{\partial \dot{q}_j}\theta_j. \tag{12.8}$$

Formula (12.8) is the n-port differential equation written in coordinate-free form. We shall now use it to write the Lagrange-Rayleigh n-port equations in Hamiltonian form.

13. HAMILTONIAN FORM OF THE LAGRANGE-RAYLEIGH n-PORT EQUATIONS

Continue with the notations of the previous section. Set:

$$p_j = \frac{\partial L}{\partial \dot{q}_j}. \tag{13.1}$$

Remark: In classical mechanics the p_j are, of course, called the momenta conjugate to the position coordinates q.

With these functions, we then have:

$$\theta(L) = p_j dq_j - Hdt, \tag{13.2}$$

with:

$$H = p_j\dot{q}_j - L. \tag{13.3}$$

It is known from our earlier work that H is a function of the variables

$$(p, q, t).$$

Recall that M denotes the space of the variables

$$(q, \dot{q}, t, v).$$

Let X be a tangent vector field to M, such that:

$$X(t) = 1.$$

Then,

$$\begin{aligned} X \lrcorner\, d\theta(L) &= X(p_j)dq_j - X(q_j)dp_j \\ &\quad - X(H)dt + dH \\ &= (X(p_j) + \frac{\partial H}{\partial q_j})dq_j \\ &\quad + (\frac{\partial H}{\partial p_j} - X(q_j))dp_j \\ &\quad + (\frac{\partial H}{\partial t} - X(H)dt. \end{aligned} \tag{13.4}$$

<u>Remark</u>: There is an inherent confusion in the notation for the partial derivatives for H. <u>As given by formula (13.3)</u>, H is a function of the variables $(q, \dot{q}, t)$. However, one

<u>proves</u> that it is a function of the p, q and t. The partial derivatives of H on the right hand side <u>then are the derivatives of H when it is expressed as a function of the variables (p, q, t)</u>. To compare the two possible sets of derivatives, it is most convenient to use the invariant definition:

$$\begin{aligned} dH &= \left(\frac{\partial H}{\partial q_j}\right)_p dq_j + \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt \\ &= \left(\frac{\partial H}{\partial q_j}\right)_{\dot{q}} dq_j + \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt \end{aligned} \tag{13.5}$$

(as in thermodynamics, $\left(\frac{\partial H}{\partial q_j}\right)_p$ denotes the "partial derivative of H, with p held fixed.")

Return to formula (13.4). Suppose that an integral curve

$$t \to \sigma(t)$$

of X satisfies relation (12.8). The differential form on the right hand side of (12.8) can be written as follows:

$$\begin{aligned} &v_j\theta_j + \frac{\partial R}{\partial \dot{q}_j}\theta_j \\ &\quad = q_j\left(v_j + \frac{\partial R}{\partial \dot{q}_j}\right)(dq_j - \dot{q}_j dt) \end{aligned} \tag{13.6}$$

Now, let us <u>suppose</u> that there are functions

$$f(p, q, t, v)$$

$$f_j(p, q, t, v)$$

such that:

$$v_j\theta_j + \frac{\partial R}{\partial \dot{q}_j}\theta_j = f_j dq_j + f dt \tag{13.7}$$

We can now put together (12.8), (13.4) and (13.7) to give the following equations, which are the Hamiltonian version of the charge-parameterized n-port Lagrange-Rayleigh equations:

$$i_j(t) \equiv \frac{dq_j}{dt} = \frac{\partial H}{\partial p_j} \tag{13.8}$$

$$\frac{dp_j}{dt} = -\frac{\partial H_j}{\partial q_j} + f_j(q(t), p(t), t, v(t)). \tag{13.9}$$

14. THE FLUX-PARAMETERIZED FORM OF THE CHARGE-PARAMETERIZED, INDUCTIVE, n-PORT

A set of n-port relations between n currents $i = (i_1, \ldots, i_n)$ and n voltages $v = (v_1, \ldots, v_n)$ is said to be of charge-controlled, inductive, type if they are of the following form:

$$v_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right), \tag{14.1}$$

where the Lagrangian function L is a FUNCTION

$$L(\dot{q})$$

of the current (i.e. $i \equiv \dot{q}$) alone. As an exercise in the use of the formalism introduced in the previous section, I shall show how these equations may be written in their "dual", flux-parameterized form.

$\theta(L)$, the Cartan form of L, is the following form:

$$\theta(L) = \frac{\partial L}{\partial \dot{q}_j} dq_j - H\, dt. \tag{14.2}$$

with: $$H = \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L. \tag{14.3}$$

Instead of "p", as in mechanics, set:

$$\varphi_j = \frac{\partial L}{\partial \dot{q}_j}. \tag{14.4}$$

Remark: The motivation for this choice is that equation (14.1) suggest that the "flux" (which should be the function whose derivative is the voltage) be set equal to $\frac{\partial L}{\partial \dot{q}_j}$.

Exercise. Show that the condition: L is independent of q: implies that: H is a function of φ_j alone. Thus,

$$\theta(L) = \varphi_j dq_j - H(\varphi)dt. \tag{14.5}$$

To solve (14.1), we must then find a curve

$$t \to \sigma(t)$$

in (q, φ, v, t)-space such that:

$$\sigma'(t) \lrcorner\, d\theta(L) = v_j \theta_j$$
$$= v_j (dq_j - \dot{q}_j\, dt). \tag{14.5}$$

Now, set:

$$i_j = \dot{q}_j. \tag{14.6}$$

(Physically, this is of course motivated by the condition that the derivative of the charge should be the current).

Using (13.5) the n-port equations (12.8) now take the following form:

$$\begin{aligned} i_j &= \frac{\partial H}{\partial \varphi_j} \\ \frac{d\varphi_j}{dt} &= v_j. \end{aligned} \tag{14.7}$$

The important point is that <u>these equations do not involve</u> q_j. To interchange the role of i and v, consider H as a "Lagrangian" function of the variables φ_j. The flux parameterized Lagrange equations then take the form:

$$i_j = -\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{\varphi}_j}\right) + \frac{\partial H}{\partial \varphi_j}$$
$$= \frac{\partial H}{\partial \varphi_j}, \text{ since H does not depend on } \dot{\varphi}_j,$$
i.e. on the voltages.

We can then sum up as follows.

<u>Theorem 14.1</u>. Consider a set of Lagrange n-port inductive,

charge parameterized relations between current and voltage variables. It can also be considered "dually" a "capacitive" Lagrangian n-port relation, with "voltage" and "current" interchanged. The Lagrangian for this dual problem is minus the <u>Hamiltonian</u> of the original problem.

This example is just the beginning of a major piece of work that needs to be done, i.e. to work out the general condition that charge-parameterized n-port relation may be "dualized" to be a flux-parameterized relation. I plan further work on this problem in a later volume.

Chapter XIII

CONTROLLABILITY AND OBSERVABILITY OF INPUT-OUTPUT SYSTEMS

1. INTRODUCTION

The controllability-observability concepts are basic to understanding the mathematical structure of systems theory. See Brockett [1], Desoer [1], Kalman, Arbib and Falb [1], and Volume VIII. The theory can only be considered wholly satisfactory, even from a theoretical point of view, in the linear case. In the non-linear case, there are differential-geometric tools available for studying the problem (particularly the Caratheday-Chow "accessibility" theorems - see DGCV) but they are not yet perfected as far as in the linear case. See the article by Sussman [1] for a brief survey of the state-of-the art, and the other articles in the Proceedings of the NATO Conference on Geometric and Algebraic Methods in Non-Linear Systems Theory, Imperial College, London, 1973, for a extensive discussion of various aspects of the problem. In this Chapter I shall present my own version of these ideas, with particular attention to the input-output systems of Lie type defined in the previous chapter.

Let us begin as in the previous Chapter. U, X, Y are real vector spaces, called the input, state, and output spaces. u, x, y denote a typical element of U, X, Y. An input is a curve

$$t \to u(t)$$

in U. The system equations are the following:

$$\frac{dx}{dt} = f(x, u, t)$$
$$y = g(x, u, t). \tag{1.1}$$

A curve $t \to (u(t), y(t))$ is an input-output pair if there is a curve $t \to x(t)$ in X such that $t \to (x(t), u(t), y(t))$ is a solution of (1.1).

The process of solving the differential equations (1.1) by the usual methods defines a map:

$$\text{(input curves)} \times \text{(states)} \to \text{(output curves)}. \tag{1.2}$$

Definition. The system (1.1) is controllable if, for every pair $(x_0, x_1) \in X \times X$ of states, every pair $(t_0, t_1) \in R$ of times, with $t_0 < t_1$, there is an input curve $t \to u(t)$ and a solution $t \to (x(t), u(t), y(t))$, $t_0 \le t \le t_1$, of (1.1) such that:

$$x(t_0) = x_0,\ x(t_1) = x_1. \tag{1.3}$$

In words, one must be able to go from one point of state space to another in given time, by appropriate choice of the control.

To define "observability," let us reformulate (1.2). It may be considered as defining a map, with the following domain and range:

$$\text{States} \to ((\text{maps of input curves to output curves})). \tag{1.4}$$

In words, the map (1.4) assigns to each $x_0 \in X$ the map

$$\varphi(x_0)\colon \text{input curves} \to \text{output curves}$$

such that the image of a curve $t \to u(t)$ under $\varphi(x_0)$ is the output curve $t \to y(t)$ such that:

$t \to (x(t), u(t), y(t))$ is the solution of (1.1), with $x(0) = x_0$.

Definition. The system (1.1) is said to be observable if the map (1.4) is one-one.

Remarks: The "controllability" and "observability" conditions are both important for practical applications. The latter is particularly relevant to what the engineers call the "system identification problem." Often, one knows the input-output relations of a system, but not the state structure. The observability condition then says that there is a way of identifying the state space via its image under the one-one map (1.4).

If the system is not observable, one can define an equivalence relation on X by saying that two elements

$$x_1, x_2 \in X$$

are equivalent if their image under the map (1.4) is the same. In other words, they are equivalent if they lie in the same fiber of the map (1.4).

Let $\hat{X}$ denote the quotient space of X under this equivalence relation. Under certain conditions, which will be investigated later, one can prove that there is an observable "quotient" system of (1.1), with input-state-output spaces $(U, \hat{X}, Y)$, and the same set of input-output pairs. This theory is due to Sussman [1].

In DGCV, Chapter 18, and in my papers, "on the accessibility problem in control theory," "Differential geometry of foliations, pt. II," "Cartan connections and the equivalence problem," "Some differential-geometric aspects of the Lagrange variational problem," I have discussed the relation between "controllability," and the Caratheodory-Chow theory of accessibility via vector field and Pfaffian systems. This work has recently been carried further by other workers in systems theory, e.g. Brockett, Hermes, Jurdjevich, Krener, Lobry, Sussman. I shall now discuss certain features of this theory.

2. CARATHEODORY-CHOW CONTROLLABILITY

Consider the system (1.1). Introduce indices and variables as follows:

$$1 \le i, j \le n = \dim X$$

$$1 \le a, b \le m = \dim U.$$

$$x = (x_i)$$

$$u = (u_a).$$

Write the first half of equations (1.1) in the following form:

$$\frac{dx_i}{dt} = f_i(x, u, t). \tag{2.1}$$

Let M be the space of variables

$$(x_i, u_a, t),$$

i.e. R^{n+m+1}, considered as a manifold.

Remark: There is an interesting bit of methodological philosophy in back of this construction. As I have emphasized before (e.g. in connection with mechanics, and thermodynamics) the idea is to define a manifold big enough to include all the relevant physical variables as real-valued functions on the manifold. Especially note that one wants to treat the "time" variable t in this way also. Physically, this means that one regards "time" as just another "observable." This is, of course, entirely consistent with the methodology of Special and General Relativity. One model for much of this is E. Cartan's treatment of the mathematics of classical mechanics in his masterpiece, "Leçons sur les invariants integraux."

Let θ_i be the 1-forms on M defined as follows:

$$\theta_i = dx_i - f_i dt. \tag{2.2}$$

Let P denote the Pfaffian system on M generated by the θ_i, i.e. the smallest F(M)-submodule of $F^1(M)$ containing the θ_i. Notice the solutions of (2.1) may be identified with the space of <u>integral curves</u> of P which are transversal to the variable "t", i.e. maps

$$\varphi: R \to M$$

such that:

$$\varphi^*(P) = 0$$

$$\varphi^*(dt) \neq 0.$$

Let V be the dual vector field system to P, i.e. the set of $Y \in V(M)$ such that:

$$\theta(Y) = 0 \tag{2.3}$$

for all $\theta \in P$.

<u>Theorem 2.1</u>. V is a free F(M)-submodule of V(M), with a basis consisting of the following vector fields:

$$Z \equiv \frac{\partial}{\partial t} + f_i \frac{\partial}{\partial x_i} \tag{2.4}$$

$$Y_a \quad \frac{\partial}{\partial u_a}; \tag{2.5}$$

<u>Exercise</u>. Prove Theorem 2.1.

<u>Definition</u>. The system (1.1) is said to be <u>Caratheoday-Chow controllable</u> if the derived system

$$V + [V, V] + [V, [V, V]] + \ldots$$

of V is all of V(M). We shall abbreviate this condition to:

<u>CC controllable</u>

<u>Remark</u>: In the general case, this condition is not <u>precisely</u> equivalent to controllability in the standard sense. The <u>possible</u> difficulty arises from the fact that among the integral curves of P are curves along which the time variable t <u>decreases</u>, while in physical and engineering problems we are only interested in curves along which time increases in

the usual way. This important subtlety has been extensively investigated by Krener, Jurdjevich and Sussman, and I will put it to the side for the moment.

Let us see what is involved in the condition that the derived system of V is all of V(M).

Set:

$$\partial_i = \frac{\partial}{\partial x_i}$$
$$\partial_t = \frac{\partial}{\partial t} \tag{2.6}$$
$$\partial_a = \frac{\partial}{\partial u_a},$$

all vector fields on M. Also, set:

$$Z_a \equiv [Y_a, Z] = \partial_a(f_i)\partial_i . \tag{2.7}$$

To verify CC controllability, we must evaluate the iterated commutators between Y_a and Z_a. I will not do this now, for the general case, but will proceed to the important cases for practical applications (see Mohler [1]), namely the <u>bilinear systems</u>.

3. CC CONTROLLABILITY FOR BILINEAR SYSTEMS

Keep the notations of Section 2. Consider the state-input relations (2.1), of the following special form:

$$\frac{dx_i}{dt} = A_{ij}x_j + B_{ia}u_a + E_{ija}x_ju_a. \qquad (3.3)$$

The coefficients (A, B, E) are given functions of t. Such a system is called a <u>bilinear state-input relation</u>.

Notice that (3.3) reduces to the well-known case of a <u>linear system</u> (considered, e.g. in Brockett [1], Desoer [1], Kalman, Arbib and Falb [1] and Volume VIII) in case the E_{ija} are zero. Thus, (3.3) is a simple generalization of the successful linear theory. Notice that it still retains certain features of "linearity," since, for a <u>given input $t \to u(t)$</u>, the differential equations (3.3) are linear. Thus, the "bilinear systems" have the feature that their analytical foundation also rests in the well-known theory of ordinary, linear differential equations.

Our program is to compute the derived system of the vector field system associated with (3.3), which is generated by:

$$Z = \partial_t + (A_{ij}x_j + B_{ia}u_a + E_{ija}x_ju_a)\partial_i \qquad (3.4)$$

$$Y_a = \partial_a.$$

Again, set:

$$Z_a \equiv [Y_a, Z] = (B_{ia} + E_{ija}x_j)\partial_i. \qquad (3.5)$$

Our first relation is:

$$[Y_a, Z_b] = 0. \qquad (3.6)$$

Write:

$$Z = \partial_t + f_i(x, u, t)\partial_i \tag{3.7}$$

$$Z_a = h_{ja}(x, t)\partial_j, \text{ with:} \tag{3.8}$$

$$h_{ia} = \partial_a(f_i). \tag{3.9}$$

Let $\mathcal{L}$ be the <u>real</u> subalgebra of V(M) generated by the Y_a, Z.

<u>Remark</u>: Note that Y_a spans a <u>nilpotent</u>, <u>abelian</u> subalgebra of $\mathcal{L}$, i.e. $(\mathrm{Ad}\ Y_a)^2 = 0$. Obviously, the theory of infinite dimensional Lie algebras will ultimately play a key role in systems theory.

For fixed $u \in U$, let K_u be the Lie algebra of vector fields on $X \times R$ generated by considering Z, Z_a, given by (3.7)-(3.8), as vector fields on $X \times R$. Note that the coefficients are linear inhomogeneous functions of x, hence the group generated by $\mathcal{K}_u$ is a group of mappings

$$R \to (\text{affine automorphisms of } X)$$

This proves the following results:

<u>Theorem 3.1</u>. A bilinear system is a Lie system, associated with the group of affine automorphisms of the vector space X.

Theorem 3.2. The bilinear system (3.3) is CC controllable if and only if, for each $u \in U$, each $(x, t) \in X \times R$,

$$\mathcal{K}_u(x) = (X \times R)_{(x,t)} \tag{3.10}$$

i.e. if the group generated by $\mathcal{K}_u$ acts in a locally transitive manner on $X \times R$.

4. INFINITESIMAL OBSERVABILITY AND THE LINEARIZED SYSTEM EQUATIONS

Return to the study of a general input-output system of form (1.1).

$$\begin{aligned} \frac{dx}{dt} &= f(x, u, t) \\ y &= g(x, u, t). \end{aligned} \tag{4.1}$$

As we have seen, there is then a map

$$X \to (\text{maps of (curves in } U) \to (\text{curves in } Y)) \tag{4.2}$$

which determines all the properties of the system. In particular, the system is said to be observable if the map (4.2) is one-one. We shall, tentatively, say that the system (4.1) is infinitesimally observable if the map (4.2), considered as a map between "manifolds," has a one-one differential.

Of course, the space on the right hand side of (4.2) is not a manifold in the usual sense. However, we shall give a sense to the definition of "infinitesimal observability" by considering the linear variational equations of the system (4.1).

Let $t \to (x(t), u(t), y(t))$ be a solution of the differential equations (4.1). Let $t \to (\delta x(t), \delta u(t), \delta y(t))$ be another curve in $X \times U \times Y$.

Definition. The functions

$$t \to (\delta x(t), \delta u(t), \delta y(t))$$

are a solution of the linear variational equations of the system (4.1), based on the given solution $t \to (x(t), u(t), y(t))$ of (4.1), if the following conditions are satisfied:

$$0 = \frac{\partial}{\partial s} \left(\frac{\partial}{\partial t} (x(t) + s\delta x(t)) - f(x + s\delta x, u + s\delta u, t)\right)/_{s=0} \tag{4.3}$$

$$0 = \frac{\partial}{\partial s} (y + s\delta y - g(x + s\delta x, u + s\delta u, t))/_{s=0} \tag{4.4}$$

Remark: The notion of "linear variational equation" may be defined (and given a beautiful geometric interpretation as the "tangent space to the set of all integral submanifolds") for arbitrary exterior differential systems. See GPS, p. 237, and my paper "E. Cartan's geometric theory of partial differential equations." Equations (4.3-(4.4) are specialization of the general equations described in these references.

Let us now adopt the index conventions used in Sections 2 and 3 to work out the specific form of equations (4.3)-(4.4).

$$x = (x_i)$$

$$u = (u_a).$$

Then, (4.2) takes the following form:

$$\frac{d}{dt}(\delta x_i(t)) = \frac{\partial f}{\partial x_i}(x(t), u(t), t)\delta x_i(t) + \frac{\partial f}{\partial u_a}(x(t), u(t), t)\delta u_a(t) \tag{4.5}$$

$$\delta y = \frac{\partial g}{\partial x_i}(x(t), u(t), t)\delta x_i(t) + \frac{\partial g}{\partial u_a}(x(t), u(t), t)\delta u_a(t). \tag{4.6}$$

Notice that this forms a linear input-output system, with $t \to \delta u(t)$ as input curve, $t \to \delta x(t)$ as state curve, and $t \to \delta y(t)$ as output. We can now make the definitive form of the "infinitesimal observability" notion:

Definition. The system (4.1) is said to be infinitesimally observable if, for each curve $t \to (x(t), u(t), y(t)$ which is a solution of the system equations (4.1), the linear, time-dependent input-output system is observable.

Remarks: Observability of linear systems like (4.5)-(4.6) has been extensively discussed in the literature. See Brockett [1]. This material will be discussed in Section 5. A key idea is that one can convert a observable linear system into a dual system which is controllable. Presumably there should be an interpretation of this dual system in terms of the non-linear system (4.1), but this has not yet been worked out.

I would also guess that the techniques of differential geometry and topology would enable one to show that "infinitesimal observability" implies some sort of "local observability," i.e. that each point of X has a neighborhood in which the map (4.2) is one-one.

5. OBSERVABILITY OF LINEAR, TIME-DEPENDENT SYSTEMS

As we have seen in Section 4, observability questions for non-linear systems may be "linearized," and reduced to linear situations. Let us now review that situation.

Let U, X, Y be finite dimensional, real vector spaces. Consider an input-output system, with U as input space, X as state space, Y as output space, of the following form:

$$\frac{dx}{dt} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u. \tag{5.1}$$

Here, $t \to A(t)$ is a one-parameter family of linear maps: $X \to X$; $t \to B(t)$: $U \to X$; $t \to C(t)$: $X \to Y$; $t \to D(t)$: $U \to Y$.

We can of course write down directly the basic system map

$$(\text{inputs}) \times \text{states} \to (\text{outputs}). \tag{5.2}$$

To this end, consider the homogeneous differential equation:

$$\frac{d\alpha}{dt} = A(t)\alpha(t). \tag{5.3}$$

It is to be solved for a curve $t \to \alpha(t) \in L(X, X)$

<u>Remark</u>: Recall our standard notation: If V_1, V_2 are vector spaces,

$$L(V_1, V_2)$$

denotes the vector space of linear maps: $V_1 \to V_2$.

Given $s \in R$, let

$$\alpha(t, s)$$

be the solution of (5.3), such that $\alpha(s, s) = 1$. In other words, the map $(t, s) \to \alpha(t, s)$ of $R^2 \to L(X, X)$ satisfies the following conditions:

$$\frac{\partial\alpha}{\partial t} = A(t) \tag{5.4}$$

$$\alpha(s, s) = 1. \tag{5.5}$$

The function $\alpha(t, s)$ satisfying (5.4), (5.5) will be called the <u>fundamental solution of the system (5.1)</u>.

Using the Variation of Constants formula of linear differential equation theory, we can now write down the explicit formula for (5.2).

$$\begin{aligned} x(t) &= \alpha(t, t_0)(x_0) \\ &+ \int_{t_0}^{t} \alpha(t, s)B(s)u(s)ds. \end{aligned} \tag{5.6}$$

<u>Exercise</u>. Show that formula (5.6) indeed gives the solution of (5.1), with initial condition:

$$x(t_0) = x_0. \tag{5.7}$$

Using (5.6), we can write the map (5.2) in the following explicit form:

$$\begin{aligned} y(t) &= C(t)\alpha(t, t_0)(x_0) \\ &+ \int_{t_0}^{t} C(t)\alpha(t, s)B(s)u(s)ds \\ &+ D(t)u(t). \end{aligned} \tag{5.8}$$

<u>Theorem 5.1</u>. The linear input-output system (5.1), is

observable with respect to a time interval

$$t_0 \le t \le t_1,$$

if and only if there is no non-zero vector $x \in X$ such that:

$$C(t)\alpha(t, t_0)(x) = 0 \quad \text{for } t_0 \le t \le t_1. \tag{5.9}$$

<u>Proof</u>. Given $x_0 \in X$, formula (5.8) determines the input → output map. Suppose the system is not observable, i.e. there are true distinct vectors x_0, $x_1 \in X$ which lead to the same input → output map. From (5.8), we see then that:

$$C(t)\alpha(t, t_0)(x_0) = C(t)\alpha(t, t_0)(x_1).$$

Setting $x = x_0 - x_1$, we have (5.8). The steps are reversible to prove the converse.

As usual, the situation is simplest if the system is time independent.

<u>Theorem 5.2</u>. Suppose that $t \to A(t)$ does not depend on the time. Then, the linear input-output system (5.1) is observable if and only if:

The intersection of the subspaces

$$\text{kernel } (C(t)A^n) \subset X \quad \text{for } t_0 \le t \le t_1,\ n = 0, 1, \ldots \tag{5.10}$$

is zero.

Proof. In this case,

$$\alpha(t, t_0) = e^{(t-t_0)A} = 1 + A(t - t_0)$$
$$+ 2A^2(t - t_0) + \dots$$

Hence,

$$C\alpha(t, t_0)x = 0$$

if and only if:

$$CA^n x = 0.$$

This proves (5.10).

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